SECTIONUM CONICARUM ELEMENTA.

SECTIONUM CONICARUM ELEMENTA

METHODO FACILLLIMA

DEMONSTRATA.

IN USUM JUVENTUTIS ACADEMICÆ.

AUTHORE L. TREVIGAR, A. M.
AULÆ CLARENSIS SOCIO.

CANTABRIGIÆ.

TYPIS ACADEMICIS.

Proſtant apud N. PREVOST, & A. VANDENHOECK & apud J. CROWNFIELD Bibliopolas LONDINENSES.

MDCCXXXI.

HONORABILI VIRO

HENRICO BROMLEY

ARMIGERO,

IN COMITATU CANTABRIGIENSI

DOMINO LOCUM-TENENTI,

ET

CUSTODI ROTULORUM;

NECNON

IN COMITIIS PARLIAMENTARIIS

PRO EODEM COMITATU

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HUMILLIME

DEDICAT

L. TREVIGAR.

PRÆFATIO.

OPUSCULUM *hoc, Lector benevole, non tam de novo confectum, quam ex aliorum scriptis, præcipue autem ex illustrissimi Marchionis de* L'HOSPITAL *tractatu excerptum senties; cum enim id mecum statuissem, ut Sectionum Conicarum Elementa clariora prodirent, & ut Philosophia* NEWTONIANA, *quantum ad Sectiones Conicas, intellectu facilior redderetur, neminem Autore nostro potiorem duxi, qui huic proposito satisfaceret; utpote qui nec longa & inutili demonstrationum serie, nec difficili Harmonicalium proportione laboraret: ideoque Theoremata maximâ ex parte ab illo desumpsi; quæque ad* NEWTONUM *intelligendum requiri & in Autore nostro desiderari videbantur, ea ab* APOLLONIO, MILNIO, *aliisque magni nominis scriptoribus adjunxi, adeo ut quæ passim & membratim inter alios erant diffusa, jam in unum quasi corpus collecta videantur.*

Theoremata quædam de circulari Sectionum curvatura in libro quinto apposui, eaque prorsus nova, apud nullos, quod scio, Sectionum Conicarum scriptores invenienda; His etiam accedit methodus proprietates omnes Parabolæ ex similibus Ellipseos vel Hyperbolæ proprietatibus derivandi; atque hic quàm latè pateat campus, quanta & quam utilia ex his principiis deduci possint, nemo, qui vel summis labris NEWTONIANAM *Philosophiam gustavit, non videt.*

Totum Autoris nostri librum sextum (de Sectionibus Conicis in Solido æstimatis) prorsus omitto; etsi enim ibi omnes Sectionum Coni-

Conicarum proprietates modo eleganti deducantur; cum tamen iſte liber nihil fere in ſe contineat, quod in prioribus non ſit demonſtratum, & cum brevitati ſimul & perſpicuitati facillimæ conſulam, ideo hanc partem non adeò neceſſariam duxi, quin tuto negligi poſſet.

Et ne brevitate nimis obſcurâ multo plus temporis in hiſce ſtudiis recolendis impenderetur, quam vel rei natura exigat, aut dignitas poſtulet, idcirco omnia quæ obſcurius & confuſius componi videbantur, ea meliori ordine diſponenda putavi, & quoad poſſem diſpoſui; non enim ut fidus interpres, verbum verbo curavi reddere, nec tamen priſtinam Autoris formam ita mutavi, ut novo prorſus cultu ornatus prodiret; at eum multo ſimpliciorem, uti ſpero, in medium profero, id ſcilicet ſtatuens, ut in Geometricâ vel mediocriter doctus, in Algebraicâ vel parum verſatus prima & præcipua Sectionum Conicarum Elementa ſine nimio ſudore percipere poſſit.

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LIBER PRIMUS DE PARABOLA

DEFINITIONES.

I.

Sit BC Regula in plano fixa; & GDO norma eâ lege posita, ut DG unum ex ejus lateribus proximè applicetur ad inferiorem Regulæ partem: sumatur filum FMO ejusdem longitudinis cum DO, altero Normæ latere: ipsius autem DO extremitati O annectatur una fili extremitas, altera verò in quovis plani puncto F, ex eâdem Regulæ parte, quà Norma; jam si Normæ latus DG per Regulæ marginem BC moveri aut labi fingas, eodem autem tempore filum paxillo aliquo M continuo intendi, dum pars ejus MO lateri DO fixa, & quasi agglutinata teneatur, Curva AMX, paxilli motu descripta, vocatur Parabola. Fig. 1.

Conversa autem Normâ super latus DO ad alteram partem puncti F, eodem modo describitur altera ejusdem Parabolæ pars AMZ, & linea XAZ erit tantum una eademque curva, quæ Parabola vocatur.

II.

Recta BC, inferior scilicet Regulæ fixæ pars, quæ & planum & Normam GDO tangit, appellatur Directrix.

III.

Punctum F in plano fixum dicitur Parabolæ Focus, vel Umbilicus.

IV.

Si a puncto F agatur recta FE ad Directricem BC normalis, Parabolæ occurrens in puncto A; recta AF indefinitè versus F producta vocatur Parabolæ axis.

V.

Recta *p*, quadrupla ipsius AF, dicitur esse axis Parameter, vel Latus Rectum.

VI.

Omnes rectæ MP, ex Parabolæ punctis ductæ ad axem normales, appellantur Ordinatim applicatæ, vel Ordinatæ ad axem.

VII.

Omnes Rectæ MO, ex Parabolæ punctis ductæ, ad axem parallelæ, vocantur Parabolæ Diametri.

VIII.

Illa Axis vel Diametri cujuscunque pars, inter verticem & Ordinatam intercepta, vocatur illius Axis vel diametri Abscissa; adeo ut AP, AP, dicantur Abscissæ.

IX. Recta.

IX.

Recta, quæ Parabolæ in uno puncto occurrit, utrinque autem producta extra cadit, dicitur Tangens in isto Parabolæ puncto.

Corollarium. I.

1. Constat ex Parabolæ definitione, si a puncto quovis M ducatur ad focum F recta MF, & ad Directricem BC normalis MD, rectas MF, MD, sibi invicem semper æquari.

Nam * $OM + MF = OM + MD$ * Def. 1.

Ablato communi OM, erit $MF = MD$

Corollarium. II.

2. Unde si ducatur recta quæcunque KK directrici BC parallela, & ex quovis Parabolæ puncto M recta MK ipsi KK normalis, & ad Focum F recta MF; KD differentia vel summa rectarum MF, MK, eadem semper *erit; differentia scilicet, quando M infra K; & summa, quando M supra K inveniatur. * 34 Elem. 1.

Corollarium. III.

3. Constat lineam FE bifariam secari in puncto A; cadente enim puncto M in A, necesse est recta MF cadat super rectam AF, recta autem MD super AE (nam punctum F est fixum) sed * MF semper æqualis est ipsi MD; unde AF ipsi AE æqualis erit. * Art. 1.

Corollarium. IV.

4. Hinc patet Methodus describendi Parabolam XAZ, ex dato axe AP (cujus vertex, A,) & ex datâ parametro p; sumantur enim super axem AP, ex utraque parte puncti A, rectæ AF, AE, singulæ æquales quartæ parti Parametri p datæ, & per punctum E ducatur recta indefinita BC ipsi FE normalis; Deinde si ponatur inferior Regulæ pars super rectam BC pro Directrice habitam, & applicetur norma GDO, ita & filum FMO lateri OD æquale, cujus altera extremitas ad punctum O, altera ad F sit fixa, & describatur Parabola XAZ, ut in prima definitione, liquet hanc fore Parabolam quæsitam:

* Def. 1. Constat etiam, quo longius fuerit normæ latus OD, hoc est, filum OMF, (ipsi OD semper * æquale) eò majorem esse Parabolæ descriptæ portionem: adeo ut pro libitu augeri vel diminui possit curva Parabolica, si & normæ latus OD, & filum OMF æqualiter augeantur vel diminuantur.

Corollarium. V.

5. Si ex quovis Parabolæ puncto M agatur ad axem Ordinata quævis MP, & ad focum F recta MF, erit $MF = AP + AF$

* Art. 1. Nam * $MF = MD = AP + AE$

* Art. 3. Sed * $AE = AF$

Unde $MF = AP + AF$

PRO-

PROPOSITIO I.

THEOREMA.

6. *Quadratum Ordinatæ cujuscunque* MP *ad axem* AP, *æquatur rectangulo sub parametro p, & Abscissâ* AP. Fig. 1.

Dico $\overline{MP}^2 = p \times AP$

Sit $AF = m$. $AP = x$. $PM = y$.

Unde * $MF = m + x$. Et $PF = m - x$, vel $x - m$; scilicet * Art. 5.
$m - x$, quando punctum P est supra, & $x - m$, quando P est infra focum F.

Jam vero,

* $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$ in utroque casu. *47. Elem. 1.

Hoc est

$$mm + 2mx + xx = yy + mm - 2mx + xx$$

Unde $yy = 4mx$.

Sed * $4m = p$ * Def. 5.

Ergo $yy = px$. Hoc est, $\overline{MP}^2 = p \times AP$.

COROLLARIUM PRIMUM ET FUNDAMENTALE.

7. MANIFESTUM est igitur, si parameter ipsius axis AP dicatur p; singulæ axis partes AP, x; Ordinatæ ejus quæcunque PM, y; semper fore $yy = px$. Et cum hæc proprietas omnibus Parabolæ punctis æquè conveniat, atque eorum situm quoad axem constanter definiat, sequitur æquationem $yy = px$ Parabolæ naturam, quoad ejus axem, perfectè & ad amussim exprimere & determinare. Fig. 2.

COROL-

Corollarium. II.

Fig. 2. 8. Si ducantur duæ Ordinatæ quæcunque MP, NQ, ad axem AP; quadrata earum erunt ad se invicem, ut axis Abscissæ AP, AQ.

Dico $\overline{MP}^2 : \overline{NQ}^2 :: AP : AQ$.

* Art. 6. Nam* $\overline{MP}^2 = p \times AP$. & $\overline{NQ}^2 = p \times AQ$

Unde $\overline{MP}^2 : \overline{NQ}^2 :: p \times AP : p \times AQ$.

* 1. Elem. 6. Sed * $p \times AP : p \times AQ :: AP : AQ$

Ergo $\overline{MP}^2 : \overline{NQ}^2 :: AP : AQ$.

Corollarium. III.

9. Hinc si Ordinatim applicatæ quotlibet æqualibus ab invicem distantiis sumantur, erunt earum quadrata in continua progressione Arithmetica: hoc est, ut numeri 1, 2, 3, 4, 5 &c. ac proinde ipsæ ordinatæ ut 1, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$ &c.

Corollarium. IV.

10. Si per punctum quodvis P in axe AP agatur recta MPM axis Ordinatis parallela, ea Parabolæ occurret in duobus punctis M & M a puncto P æque remotis; rectæ enim MP, MP, cum ponantur esse ad axis Ordinatas parallelæ, Parabolæ occurrunt (ei enim occurrunt Ordinatæ ipsæ) & ad axem *erunt normales, ideoque ad Axem Ordinatæ. Unde singula quadrata * Def. 6.

$\overline{MP}^2$, $\overline{MP}^2$, eidem rectangulo px æquantur, & $MP = MP$; hoc est, puncta, M & M æqualiter distant a puncto P.

Corol-

COROLLARIUM. V.

11. CUM * sit $yy = px$. vel $\overline{MP}^2 = p \times AP$, liquet, quo major fuerit AP vel x, eo majorem fore PM vel y, ex utraque axis parte sumptam: adeo ut si AP in infinitum producatur, PM infinitè etiam augebitur. Et è contrario, quo minor fuerit AP, eò minor erit Ordinata PM; ideoque si AP fiat infinitè parva, vel evanescat, PM ex utraque axis parte erit quoque infinitè parva vel evanescet: hoc est, si punctum P cadat super punctum A, puncta M & M sibi mutuo occurrent in P. unde patet, * Art. 7.

1mo. RECTAM LL per axis verticem A ductam, & ad Ordinatas parallelam, esse tangentem in puncto A.

2do. SINGULA Parabolæ puncta eo longius ab axe ejus AP distare, quo remotiora sint a vertice A; ideoque recta LM, axi AP parallela, Parabolæ non nisi in uno puncto occurrit, distantia enim ejus ab axe eadem semper manet, a Parabolâ autem continuo augetur.

COROLLARIUM. VI.

12. SI ex quovis Parabolæ puncto M agatur recta ML axi AP parallela, rectæ autem AL Ordinatis parallelæ occurrens in L, patet esse * $AL = PM = y$. Ob eandem rationem $ML = AP = x = \frac{yy}{p}$ * 34. Elem. 1.

(nam * $px = yy$ & dividendo per p, erit $x = \frac{yy}{p}$.) Unde sequitur rectas ML, ML, ex utraque axis parte sumptas, tum sibi invicem æquari, cum puncta L & L æqualiter distant a puncto A; nam ex istâ Hypothesi $LA = LA$, hoc est $PM = PM$, ac proinde $\frac{yy}{p} = \frac{yy}{p}$ (nam p quantitas eadem est & constans) ergo $ML = ML$. * Art. 7.

COROL-

COROLLARIUM. VII.

13. SI recta MM, ad Parabolam utrinque terminata, bifariam secetur ab axe AP, ea tangenti LL parallela, erit; ductis enim ML, ML, axi AP parallelis, liquet rectam LL bifariam secari in puncto A, quia MM bisecatur in
* Art. 12. P: unde *ML, ML, sibi invicem æquantur; ideoque
* 33. El. 1. *erit MM ipsi LL parallela. Est etiam MM ordinata
* Art. 11. ad axem, recta enim LL *parallela est Ordinatis ad Axem: sed & rectæ MM est parallela, unde & ipsa MM erit Ordinata.

COROLLARIUM. VIII.

14. CUM omnes perpendiculares MPM ab axe AP bifariam secentur, constat Parabolam ipsam ab axe dividi in duas partes æquales, similes, & similiter positas; si enim plani Parabolici pars una in alteram superimponatur, hoc est, si omnes Ordinatæ MP ponantur super omnes Ordinatas PM, hæ illis perfectè congruent, ac proinde tota Parabolæ portio ex una axis parte coincidet cum tota parabolæ portione ex altera axis parte, cum portiones istæ ex infinito Ordinatarum numero componantur & constituantur.

LEMMA. I.

Fig. 3. 15. SINT ABC, DEF *duo triangula, quorum latera sibi mutuo parallela fuerint; videlicet* AB *ipsi* DE, AC *ipsi* DF, BC *ipsi* EF, *dico hæc triangula esse similia.*

* 10. Elem. 11. OB latera AB, BC ipsis DE, FE parallela, * erit angulus ABC=DEF: ob eandem rationem angulus BAC =EDF, ita & angulus ACB=DEF: unde triangula
*4. Elem. 6. *erunt similia.

PRO-

PROPOSITIO II.

THEOREMA.

16. SI *per axis* AP *verticem* A *ducatur in utrovis angulorum* PAL, PAL, *sub axe* AP *&* *rectâ* LL *ordinatis parallela contentorum, recta quæcunque* AM; *dico hanc Parabolæ* MAM *iterum occurrere in alio puncto* M. Fig. 4.

SUMPTA enim super AL, ex utravis parte puncti A, rectâ AG axis parametro p æquali; actâque GF axi AP parallelâ, quæ ipsi AM (productæ si opus) in puncto F occurrat; capiatur super rectam AL (ex eadem axis parte, qua AM) pars AL ipsi GF æqualis; & fiat LM axi AP parallela; dico punctum M, ubi LM ipsi AM occurrit, esse ad Parabolam.

SIT enim MP ipsi AL parallela, & triangula FGA, APM erunt * similia; ac proinde si fiat PM$=y$, AP$=x$, GA$=p$. erit FG, vel AL, vel PM : GA : : AP : PM, Hoc est, $y:p::x:y$. Unde * $yy=px$; PM igitur erit Ordinata * ad axem, adeoque punctum M est ad Parabolam. * Art. 15. * 16. El. 6. * Art. 7.

COROLLARIUM I.

17. HINC ex datis Parabolæ axe AP, & ejus parametro p, ductâque per verticem A rectâ AM in utrovis angulorum PAL, PAL, facilè innotescit punctum M, ubi recta AM Parabolæ occurrit: Ducatur scilicet recta AG, parametro datæ p æqualis, & recta GF, axi AP parallela; Productâ GF, usque dum ipsi AM occurrat in F, fiat AL$=$GF, & LM ipsi GF parallela; liquet igitur punctum M, ubi recta LM ipsi AM occurrit, * esse punctum quæsitum. * Art. 16.

COROLLARIUM II.

18. MANIFESTUM est, nullam rectam præter LAL Parabolam tangere posse in puncto A; ea enim est unica, quæ, per verticem A producta, tota cadat extra Parabolam; quamcunque enim duxeris ab LAL diversam, ea in angulo PAL necessario cadet, ac proinde Parabolæ in alio puncto præter A * occurrit.

* Art. 16.

DEFINITIONES.

X.

Fig. 5 & 6. SI per quodvis Parabolæ punctum M agantur Diameter MO, Ordinata MP ad axem AP, & recta MT, quæ ab axe AP, ultra verticem ejus producto, auferat partem AT ipsi AP æqualem; omnes rectæ NO, ductæ ex Parabolæ punctis quibuscunque ipsi MT parallelæ, & ad Diametrum MO terminatæ, vocantur ORDINATÆ, vel ORDINATIM APPLICATÆ AD DIAMETRUM MO.

XI.

SI fiat recta q tertia proportionalis ad ipsas AT, MT, ista recta q dicitur PARAMETER diametri MO.

COROLLARIUM I.

19. FIAT AP, vel AT $= x$. & erit $\overline{MT}^2 = qx$.

* Def. 11. Nam * AT : MT : : MT : q.

Hoc est x : MT : : MT : q.

* 17. El. 6. Ergo * $\overline{MT}^2 = qx$.

COROL-

COROLLARIUM II.

20. PARAMETER q diametri cujuſcunque MO, excedit parametrum p axis AP quantitate $4x$, quadrupla axis AP (x).

Nam * $\overline{MT}^2 = \overline{MP}^2 + \overline{PT}^2$. * 47. El. 1.
Hoc eſt * $qx = px + 4xx$. * Art. 19.
Unde $q = p + 4x$.

COROLLARIUM III.

21. PARAMETER q diametri cujuſcunque MO, æquatur quadruplæ rectæ MF, ab illius diametri origine M ad focum F ductæ.

Scilicet $q = 4MF$.
Nam * $q = p + 4x$, * Art. 20.
Sed * $p = 4AF$, & $4x = 4AP$. * Def. 5.
Ergo $q = 4AF + 4AP$,
Sed * $4MF = 4AF + 4AP$, * Art. 5.
Unde $q = 4MF$.

PROPOSITIO III.

THEOREMA.

22. QUADRATUM *Ordinatæ cujuſvis* ON *ad diametrum* MO, *æquatur rectangulo ſub parametro q, & abſciſſâ* MO.

Dico $\overline{ON}^2 = q \times MO$.

DUCTA ad axem AP ordinata NQ diametro MO in puncto R occurrat; & ſit OH ipſi MP parallela. Fig. 5. & 6.

Fiat AP vel AT $= x$. PM vel RQ $= y$. OR vel HQ $= a$. MO vel PH $= b$.

* Art. 15. Jam triangula TPM, ORN, sunt * similia.

Unde TP : PM : : OR : RN.

Hoc est, $2x : y : : a : \frac{ay}{2x} = RN$.

His positis, erit in Fig. 5^ta^.

$NQ = RQ - RN = y - \frac{ay}{2x} = \frac{2xy - ay}{2x}$,

Vel $NQ = RN - RQ = \frac{ay}{2x} - y = \frac{ay - 2xy}{2x}$,

In utroque casu, $\overline{NQ}^2 = yy - \frac{ayy}{x} + \frac{aayy}{4xx}$.

Porro in Fig. 6^ta^.

$NQ = RQ + RN = y + \frac{ay}{2x}$,

Et $\overline{NQ}^2 = yy + \frac{ayy}{x} + \frac{aayy}{4xx}$,

Unde in figuris 5^ta^ & 6^ta^ erit,

$\overline{NQ}^2 = yy \mp \frac{ayy}{x} + \frac{aayy}{4xx}$.

Rursus in Fig. 5^ta^ erit,

$AQ = AH - HQ = AP + PH - HQ = x + b - a$.

In Fig. 6^ta^.

$AQ = AH + HQ = AP + PH + HQ = x + b + a$,

Ideoque in figuris 5^ta^ & 6^ta^, erit $AQ = x + b \mp a$,

* Art. 8. Jam vero, *$AP : AQ : : \overline{PM}^2 : \overline{QN}^2$:

Hoc est, $x : x + b \mp a : : yy : yy + \frac{byy}{x} \mp \frac{ayy}{x}$.

Unde $\overline{QN}^2 = yy + \frac{byy}{x} \mp \frac{ayy}{x}$;

Sed

Sed $\overline{QN}^2 = yy \mp \frac{ayy}{x} + \frac{aayy}{4xx}$.

Ergo $yy \mp \frac{ayy}{x} + \frac{aayy}{4xx} = yy + \frac{byy}{x} \mp \frac{ayy}{x}$;

Auferendo utrinque $yy \mp \frac{ayy}{x}$, dividendo per yy, & ducendo in $4xx$, erit $aa = 4bx$,

Scilicet $\overline{OR}^2 = 4bx$;

Sed triangula MPT, NRO sunt *similia, *Art. 15.

ac proinde $\overline{PT}^2 : \overline{OR}^2 :: \overline{MT}^2 : \overline{ON}^2$,

Hoc est, $4xx : 4bx :: {}^*qx : \frac{4bqxx}{4xx} = bq$. *Art. 19.

Ergo $\overline{ON}^2 = bq = q \times MO$.

COROLLARIUM I.

23 MANIFESTUM est, quæcunque demonstrata fuerint in propositione prima de axe AP, locum habere apud omnes diametros MO, earum Ordinatas ON, & parametrum q. Et cum articuli 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, a prima propositione deducantur, verique sint, utrum anguli APM recti fuerint, necne; sequitur, si fingamus rectam AP non esse axem, sed diametrum quamcunque, cujus Ordinatæ sint PM, NQ, & parameter p, hos articulos adhuc veros fore, eorum enim demonstratio prorsus eadem erit.

COROL-

COROLLARIUM II.

Fig. 5 & 6. 24. CUM articuli 11 & 18 æqualiter obtineant, sive AP sit axis, sive Diameter quæcunque ut MO, constat rectam MT, diametri Ordinatis parallelam, esse tangentem in puncto M, nullamque aliam rectam in illo puncto Parabolam tangere posse.

UNDE patet, unam solummodo tangentem duci posse per datum Parabolæ punctum.

COROLLARIUM III.

25. HINC, si per quodvis Parabolæ punctum M agatur ad axem ordinata quævis MP, & recta MT, quæ ab axe, ultra verticem producto, auferat partem AT ipsi AP æqualem, recta MT erit contingens in puncto
* Def. 10. M; nam MT parallela *est Ordinatis ad Diametrum,
* Art. 24. ac proinde Parabolam *tangit in puncto M.

ET vice versâ, si recta MT Parabolam in puncto M tangat, & ad axem ducatur Ordinata MP, partes AT, AP, erunt æquales; non enim: sed si fieri potest, non sit AT ipsi AP æqualis; liquet igitur rectam aliquam duci posse per punctum M, quæ auferat partem, ipsi AP æqualem; & hæc erit tangens: unde duæ rectæ Parabo-
* Art. 24. lam tangunt in eodem puncto M, *quod fieri nequit.

COROLLARIUM IV.

Fig. 7. 26. SI in definitionibus 10 & 11, & in ultima hac propositione, ponamus rectam AP non esse axem, sed diametrum quamcunque, cujus Ordinatæ sint PM, QN, adhuc constabit hujus propositionis veritas, cum eadem prorsus sit demonstratio, ut ex sola figuræ septimæ inspectione

spectione patet; ex similibus enim triangulis TPM, ORN, eadem elicitur proportio, ac cum AP pro axe habebatur.

Unde si ex Parabolæ quovis puncto M agatur ad Diametrum AP ordinata MP, ita & recta quædam MT, eandem AP ita secans in T, ut sit AT=AP, erit MT tangens in puncto M, & vice versâ.

Liquet etiam, Diametrum MO (ex hac Hypothesi) pro axe ipso haberi posse; ideoque axis pro diametro habendus, quæ cum Ordinatis ejus faciat rectos angulos.

PROPOSITIO IV.

Theorema.

27. Ductis *ad Diametrum quamcunque* AP *Semi-ordinatis* MP, NQ; *dico parametrum* p *esse ad summam Semi-ordinatarum* MP, NQ, *ut differentia illarum ad differentiam earundem abscissarum* AP, AQ. Fig. 8.

Scilicet $p : MP + NQ :: NO : MO$.

$$\text{Nam } {}^{*}AQ = \frac{\overline{NQ}^2}{p} \text{ Et } AP = \frac{\overline{MP}^2}{p},$$ * Art. 6. & 22.

$$\text{Ergo } AQ - AP = \frac{\overline{NQ}^2 - \overline{MP}^2}{p};$$

$$\text{Unde } \overline{AQ - AP} \times p = \overline{NQ}^2 - \overline{MP}^2.$$

$$\text{Sed } \overline{NQ}^2 - \overline{MP}^2 = \overline{NQ + MP} \times \overline{NQ - MP},$$

$$\text{Ergo } \overline{AQ - AP} \times p = \overline{NQ + MP} \times \overline{NQ - MP};$$

$$\text{Vel, } {}^{*}p : NQ + MP :: NO : MO.$$ * 16. El. 6.

PROPOSITIO V.

THEOREMA.

Fig. 9. 28. *Si per quodvis Parabolæ punctum* M *ducatur ad axem Ordinata* MP, *& ad tangentem, per punctum* M *ductam, demittatur normalis quædam* MG; *dico axis partem* PG *dimidiæ parametri p semper esse æqualem.*

Hoc est, $PG = \frac{1}{2}p$.

* Def. 6. Quoniam MP ipsi TG normalis * sit, erunt triangula TPM, GPM * similia, ac proinde

* 8. Elem. 6.

$$TP : PM :: PM : PG$$

Hoc est, $2x : y :: y : \frac{yy}{2x}$ unde $PG = \frac{yy}{2x}$;

* Art. 7. Sed * $yy = px$. Ergo $PG = \frac{px}{2x} = \frac{1}{2}p$.

PROPOSITIO VI.

THEOREMA.

Fig. 9. 29. *Si ab axis* AP *vertice* A *erigatur usque ad quamlibet contingentem* MT *perpendicularis* AK; *& a puncto* K *erecta ad contingentem perpendicularis* KF *secet axem in* F; *erit ubique* AF *æqualis quartæ parti Parametri axis.*

Scilicet $AF = \frac{1}{4}p$.

* Art. 25. A tactu M ordinatâ MP ad Axem AP, erit * $AP = AT$.

* 2. El. 6. Unde * $AK = \frac{1}{2}MP$, & $\overline{AK}^2 = \frac{1}{4}\overline{MP}^2$.

* 8. El. 6. Sed * $\overline{AK}^2 = AF \times AT$,

Et

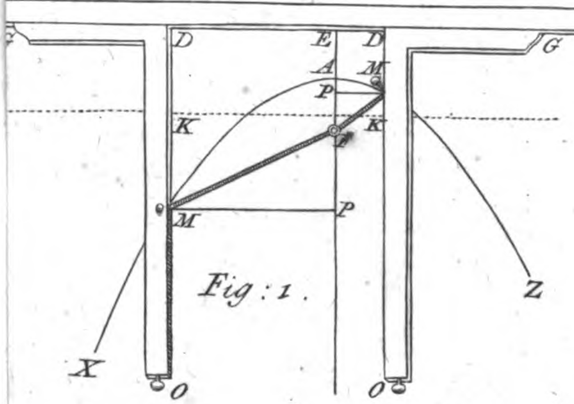


Fig: 1.

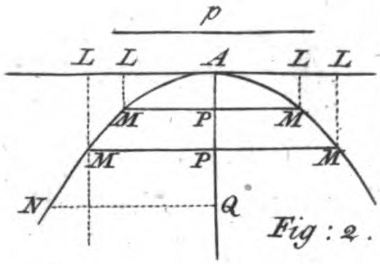
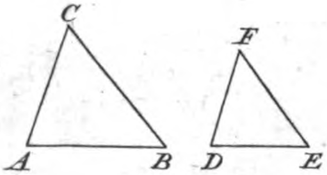
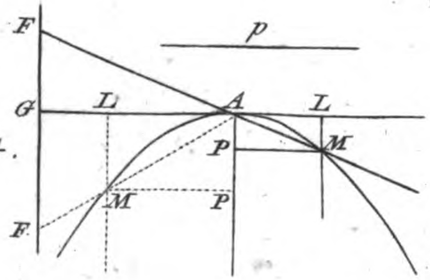


Fig: 2.





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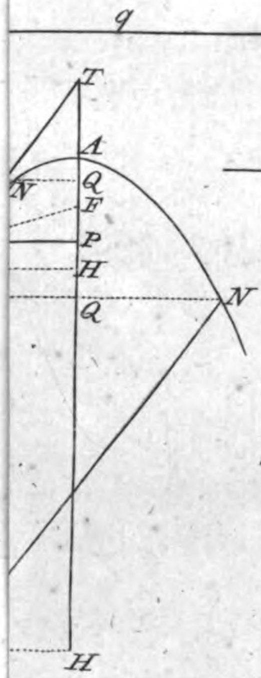


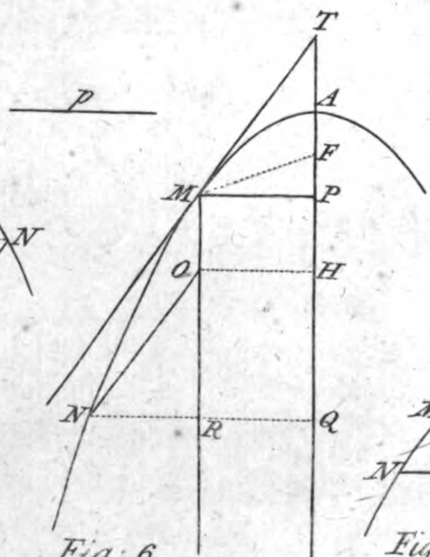


Fig: 6.

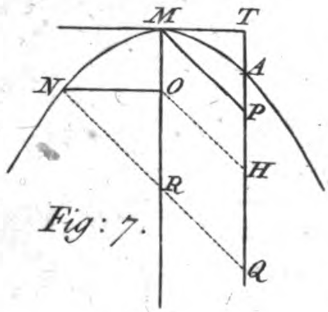


Fig: 7.

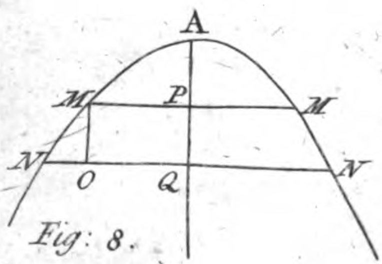


Fig: 8.

Et * $\frac{1}{4}\overline{MP}^2 = AP \times \frac{1}{4}p = AT \times \frac{1}{4}p$. * Art. 6.

Ergo $AF \times AT = AT \times \frac{1}{4}p$,

Hoc est, $AF = \frac{1}{4}p$.

COROLLARIUM I.

30. HINC constat, * punctum F esse Parabolæ FOCUM. * Def. 3. & 5.

COROLLARIUM II.

31. RECTA HF, per focum ducta, & ad axem Ordinata, æquatur semiparametro p; nam * $AF = \frac{1}{4}p$, unde * erit $\overline{HF}^2 = \frac{1}{4}p \times p = \frac{1}{4}pp$. Ergo $HF = \frac{1}{2}p$. * Art. 29. * Art. 6.

PROPOSITIO VII.

THEOREMA.

32. SI *per quodvis Parabolæ punctum* M, *agatur ad Focum* F *recta* MF, *& ex eodem puncto diameter quævis* MO, *& tangens* TMS; *dico angulum* FMT, *ab una tangentis parte* TM, *& recta* MF *factum, æquari angulo* OMS, *sub alterâ tangentis parte* MS, *& Diametro* OM *comprehenso.*

Scilicet $FMT = OMS$.

Producatur Axis AP, donec tangenti TMS occurrat in T, & sit MP ordinata ad axem: unde erit * $TA = AP$, * Art. 25.

Addatur utrinque AF,

Et erit $TA + AF = AP + AF$,

Sed * $MF = AP + AF$, * Art. 5.

Unde $TF = MF$,

* 5. El. 1. Et ang. * $FMT = FTM$;
* 27. El. 1. Sed ang. * $FTM = OMS$,
Ergo $FMT = OMS$.

COROLLARIUM I.

33. HINC si a Foco F demittatur normalis FK ad
* 26. El. 1. tangentem MT, illa tangentem bifariam * secabit in puncto K.

COROLLARIUM II.

34. UNDE $MF \times AF = \overline{FK}^2$; nam ob triangulum MFT Isosceles, & demissam perpendicularem FK, erit
* 4. El. 6. angulus $TFK = KFM$, & triangulum AFK * simile triangulo KFM; ergo erit $AF : FK :: FK : MF$, ideoque $MF \times AF = \overline{FK}^2$.

PROPOSITIO VIII.

PROBLEMA.

Fig. 10. 35. DATIS *Diametro, & tangente per verticem ejus ductâ, ita & parametro ejus; Parabolam ipsam motu continuo describere.*

SI diameter data esset axis, facile describi posset Parabola secundum articulum quartum; sin minus, sit MO diameter data, & TMS tangens per verticem ejus ducta.

SUPER

SUPER Diametrum ultra verticem M productam, sumatur pars MD æqualis quartæ parti Parametri ejus datæ: per punctum D agatur DE ipsi MD normalis, ducaturque MF ita, ut angulus FMT æqualis sit angulo OMS; factâ deinde MF ipsi MD æquali, describatur, ut in primâ definitione, Parabola, cujus directrix sit DE, & Focus F; dico hanc fore Parabolam quæsitam.

RECTA enim MO, cum directrici DE normalis sit, axi parallela erit, ideoque etiam *diameter: Recta autem * Def. 7.
TMS Parabolam *tangit in puncto M; & Parameter * Art. 32.
diametri MO *est quadrupla ipsius MF. * Art. 21.

PROPOSITIO IX.

PROBLEMA.

36. DATIS *Diametro* AP, *Parametro ejus, & tangente* AL, *per istius diametri verticem* A *ductâ; Parabolam per plurima puncta describere.* Fig. 11.

DUCTA, per tangentis AL punctum quodvis L, rectâ indefinita LE diametro AP parallelâ; sumantur super LE & super diametrum AP ultra verticem ejus productam, partes LE, EE, EE, *&c.* AF, FF, FF, *&c.* sibi mutuo æquales, cujusvis autem magnitudinis. Notetur super LE punctum M, ita ut LM sit tertia proportionalis ad parametrum Diametri AP datam, & ad partem AL tangentis datæ. Ductis igitur per puncta A & M rectis AE, AE, AE, *&c.* MF, MF, MF, *&c.* dico puncta intersectionis, N, N, N, *&c.* esse ad Parabolam quæsitam.

ACTIS enim per punctum notatum M & per punctum aliquod inventum N, rectis MP, NQ, tangenti AL parallelis; fiat $AP = x$. PM vel $AL = y$. $AQ = v$. $NQ = z$.

* Art. 15. Jam vero Triangula NQA, ALE *sunt similia; ita
* 2 Elem. 6. & *MPF, NQF. Unde erit,

NQ : QA :: AL : LE,

Hoc est, $z : v :: y : \frac{vy}{z}$, Unde LE vel AF $= \frac{vy}{z}$.

Rursus MP : PF vel PA + AF :: NQ : QF vel QA + AF,

Hoc est, $y : x + \frac{vy}{z} :: z : v + \frac{vy}{z}$;

* 16. El. 6. Ergo *$vy + \frac{vyy}{z} = xz + vy$; deleto utrinque vy, &

erit $\frac{vyy}{z} = xz$, vel $vyy = xzz$.

* 16. El. 6. Unde *$x : v :: yy : zz$.

Hoc est, AP : AQ :: $\overline{MP}^2 : \overline{NQ}^2$.

*Construct. Jam vero quadratum ex AL vel PM *æquatur rectangulo sub diametri parte AP, & Parametro ejus; unde
* Art. 22. *erit PM ordinata ad diametrum AP, ideoque etiam QN; Punctum igitur N erit in Parabola ex alterutra parte diametri AP. Ideoque ad invenienda puncta ex altera parte istius diametri, capiantur, super rectas indefinitas LE, AF, partes æquales LE, EE, &c. AF, FF, &c. ex altera parte punctorum L & A.

LIBER

LIBER SECUNDUS DE ELIPSI

DEFINITIONES.

I.

SUMATUR filum FMf; & in plano quovis affigantur ejus extremitates ad puncta F, f, quorum distantia ab invicem longitudine fili minor sit; deinde ope paxilli M continuo tensum teneatur filum, & eodem tempore circumagatur paxillus circa puncta F, f, donec in orbem rediens ad idem punctum, unde digressus est, iterum revertatur. Linea autem paxilli motu descripta, vocatur ELLIPSIS. Fig. 12. & 13.

II.

PUNCTA fixa F, f, vocantur Ellipseos FOCI, seu UMBILICI.

III.

RECTA Aa per focos ducta, & ab Ellipsi utrinque terminata, dicitur PRIMUS vel MAJOR AXIS, vel LATUS TRANSVERSUM, vel AXIS PRINCIPALIS, vel TRANSVERSUS.

IV.

PUNCTUM C, ubi primus axis Aa bifariam secatur, appellatur Ellipseos CENTRUM.

RECTA

V.

Recta Bb per Ellipseos centrum ducta ad primum axem Aa perpendicularis, & ab Ellipsi utrinque terminata, dicitur Secundus, vel Minor Axis.

VI.

Axes Aa, Bb, simul sumpti appellantur Conjugati; adeo ut primus axis Aa dicatur esse secundo Bb Conjugatus; & vicissim secundus Bb primo axi Aa Conjugatus.

VII.

Rectæ MP, MK, per Ellipseos puncta, ut M, ductæ, uni axium parallelæ, & ab altero terminatæ, dicuntur esse ad hunc alterum Ordinatæ; adeo ut MP sit ad Axem Aa Ordinata, & MK ad Axem Bb.

VIII.

Tertia proportionalis ad duos axes, vocatur Parameter primi in proportione termini: si igitur fiat, ut axis Aa ad axem Bb, ita axis Bb ad tertiam p, tertia illa p erit axis Aa Parameter.

IX.

Omnes rectæ per Centrum C ductæ, & ab Ellipsi utrinque terminatæ, Diametri appellantur.

X.

Recta, quæ Ellipsi in uno tantum puncto occurrit, utrinque

utrinque autem producta extra cadit, TANGENS est in illo Ellipseos puncto.

XI.

ILLA axis vel diametri cujusvis pars, inter Ordinatam & verticem comprehensa, vocatur ABSCISSA.

SCHOLIUM.

37. SI puncta F, f, in unum coeant ad centrum C, constat Ellipsin mutatam iri in circulum, cujus radius dimidio fili CMC æqualis sit; adeo ut circulus pro Ellipsi haberi possit, in quâ focorum distantia evanescat; quæcunque igitur in sequentibus de Ellipsi fuerint demonstrata, etsi maxima fuerit focorum ab invicem distantia, de Circulo tamen æquè vera sunt, & æqualiter obtinent, si modo ponamus distantiam hanc evanescere vel nullam fieri. Fig. 14.

COROLLARIUM I.

38. CONSAT ex primâ definitione, si ex quovis Ellipseos puncto M agantur ad focos F, f, rectæ MF, Mf, earum summam semper esse eandem, utpote quæ fili ipsius longitudini semper æqualis sit. Fig. 12. & 13.

COROLLARIUM II.

39. CADENTE puncto M in A, necesse est recta MF cadat super rectam AF, ita & recta Mf super Af. Fig. 12.
Ob eandem rationem, puncto M in a cadente, recta MF cadit super aF, & Mf super af. Unde patet esse, Fig. 13.
$AF + Af = MF + Mf$.

Hoc

Hoc eſt, $2AF + Ff = MF + Mf$,

Sed $2af + Ff = Mf + MF$ ob eandem rationem.

Ergo $2AF + Ff = 2af + Ff$. Unde $AF = af$.

Hinc ſequitur 1*mo*, ſummam rectarum, MF, Mf, axi majori Aa ſemper æquari.

Nam $MF + Mf = AF + Af$,

Sed $AF = af$,

Ergo $MF + Mf = Af + af = Aa$.

2*do*. Liquet ſocorum diſtantiam Ff biſecari a Centro C,

* Def. 4. Nam * $CA = Ca$; & $AF = af$,

Unde $CA - AF = Ca - af$ hoc eſt, $CF = Cf$.

Corollarium III.

Fig. 12. 40. Si a termino B axis ſecundi Bb ducantur ad duos

* Def. 5. focos F, f, rectæ BF, Bf, patet triangula * rectangula BCF, BCf, eſſe æqualia; nam anguli ad C ſunt recti, & latera BC, CF, æquantur lateribus BC, Cf, unde

* 4. El. 1. * $BF = Bf$.

Hinc BF, vel $Bf = CA$ vel Ca,

Nam $BF + Bf = CA + Ca$.

Fig. 13. Eodem modo oſtenditur eſſe,

Fb vel $bf = CA$ vel Ca. Unde conſtat,

1*mo*. Axem ſecundum Bb biſecari in C;

* 47. El. 1. Nam $BF = bF$. Unde * $\overline{BC}^2 + \overline{CF}^2 = \overline{bC}^2 + \overline{CF}^2$,

Ergo $\overline{BC}^2 = \overline{Cb}^2$, & $BC = Cb$.

2*do*. Axem ſecundum Bb primo Aa ſemper eſſe mi-

* 47. El. 1. norem; nam BC * minor eſt quàm BF, hoc eſt, quàm CA.

3*tio*. Si a termino B axis ſecundi Bb, tanquam centro, deſcribatur circulus radio BF ipſi CA æquali, iſte circulus

culus axem primum Aa in ipsis Ellipseos focis F, f, secabit; rectæ enim, a puncto B ad focos F, f, ductæ, ipsi CA sunt æquales: sed radius circuli æquatur & ipsi CA: unde rectæ, a puncto B ad focos F, f, ductæ, ad circuli circumferentiam usque pertingent: circulus igitur descriptus per focos transit.

COROLLARIUM IV.

41. HINC ex datis axibus Aa, Bb, facilè describi potest Ellipsis; * inventis enim focis F, f, annectatur filum ejusdem longitudinis cum Axe Aa, & describatur Ellipsis, ut in primâ definitione. * Art. 40.

COROLLARIUM V.

42. QUADRATUM dimidii axis secundi Bb, æquatur rectangulo sub AF & Fa, partibus axis primi, inter focum F, & terminos A, a, comprehensis.

Dico $\overline{BC}^2 = AF \times Fa$.

Sit CA vel * BF $= t$: CF $= m$. * Art. 40.

Et erit * $\overline{BC}^2 = \overline{BF}^2 - \overline{CF}^2 = tt - mm$, * 47. El. 1.

Sed $AF = AC - CF = t - m$,

Et $Fa = AC + CF = t + m$,

Unde $AF \times Fa = tt - mm = \overline{BC}^2$.

LEMMA II.

43. SICUT *Quatuor proportionales in æquationem facillimè convertuntur, ex eò quòd rectangulum sub extremis* * *æ-* * 16. El. 6.

 quetur

quetur rectangulo sub mediis; ita vicissim æquatio ad proportionem reducitur, habitis magnitudinibus ex unâ æquationis parte pro extremis, ex alterâ autem pro mediis.

Nam facta ex utrâque æquationis parte pro æqualibus rectangulis haberi possunt; & hæc rectangula latera
* 14. El. 6. habent *reciproca: istæ autem quantitates, quæ inter se ductæ factum ex alterutrâ æquationis parte conficiunt, pro rectanguli lateribus sunt habendæ; unde termini in proportione inveniendâ innotescent.

PROPOSITIO I.

Theorema.

Fig. 12 & 13. 44. Si *ad axem primum* A*a* *ducatur Ordinata* MP, *& ab eodem axe auferatur pars* AD, *ipsi* MF *æqualis; erit* CA : CF :: CP : CD.

Sit CA $= t$. CF $= m$. CP $= x$. PM $= y$. CD $= z$.

Fig. 12. Cadat Primò punctum P supra centrum C;

Erit igitur PF semper minor, quàm P*f*;

* 47. El. 1. Sed * $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$,

Et $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$,

Unde MF vel AD minor, quàm M*f* vel *a*D;

Ac proinde AD seu MF $=$ AC $-$ CD $= t - z$.

Et *a*D vel M*f* $=$ *a*C $+$ CD $= t + z$.

PF $=$ CF $-$ CP, vel CP $-$ CF $= m - x$ vel $x - m$, prout punctum P cadat infra vel supra focum F.

Erit etiam P*f* $=$ CP $+$ C*f* $= x + m$.

Jam verò, $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$.

Hoc

Hoc eſt, $tt - 2tz + zz = yy + mm - 2mx + xx$;

Sed & $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$,

Hoc eſt, $tt + 2tz + zz = yy + mm + 2mx + xx$; ac proinde, ſi membra prioris Æquationis reſpectivè ſubducantur ex membris poſterioris, reſidua erunt æqualia;

Scilicet $4tz = 4mx$,

Et $tz = mx$,

Unde * $t : m :: x : z$, vel $\frac{mx}{t}$, * Art. 43.

Hoc eſt, $CA : CF :: CP : CD = z$ vel $\frac{mx}{t}$.

2do. CADAT Punctum P infra C, & erit ſemper PF major quam Pf; unde MF vel AD major, quàm Mf vel aD, ideoque erit Fig. 13.

AD ſive MF $=$ AC $+$ CD $= t + z$;
aD ſive $Mf = a$C $-$ CD $= t - z$;
PF $=$ PC $+$ CF $= x + m$;
Pf $=$ PC $-$ Cf, vel Cf $-$ CP $= x - m$, vel $m - x$, prout punctum P cadat infra vel ſupra focum f;

Eſt autem $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$,
Hoc eſt, $tt + 2tz + zz = yy + mm + 2mx + xx$,
Et $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$.

Scilicet, $tt - 2tz + zz = yy + mm - 2mx + xx$, ac proinde ſi membra poſterioris æquationis reſpectivè ſubducantur ex membris prioris, reſidua erunt æqualia;

Scilicet $4tz = 4mx$,

Et $tz = mx$;

Ergo * $t : m :: x : z$, vel $\frac{mx}{t}$, * Art. 43.

Hoc eſt, $CA : CF :: CP : CD = z$ vel $\frac{mx}{t}$.

 COROL-

COROLLARIUM.

Fig. 12. 45. HINC, si fiat CA vel $Ca = t$, CF vel $Cf = m$; semper erit $MF = AC - CD = t - \frac{mx}{t}$, ita & $Mf = aC + CD = t + \frac{mx}{t}$, in primo casu, quando P scilicet supra centrum C cadit.

Fig. 13. Cadente autem P infra C, erit semper $MF = AC + CD = t + \frac{mx}{t}$. Et $Mf = t - \frac{mx}{t}$.

PROPOSITIO II.

THEOREMA.

Fig. 12, & 13. 46. QUADRATUM *Ordinatæ Cujusvis* MP *ad axem* Aa, *est ad rectangulum sub abscissis* AP, Pa, *ut ejus conjugati* Bb *quadratum, ad ipsius* Aa *quadratum*.

Dico $\overline{MP}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

Sic $CA = t$. $CF = m$. $CP = x$. $PM = y$. $CD = z$.

Et erit $MF = AC - CD$, vel $AC + CD = t - z$, vel $t + z$, prout punctum P supra vel infra centrum C cadat.

Unde erit $\overline{MF}^2 = tt \mp 2tz + zz$.

Porro, $PF = CF - CP$, vel $CP - CF$, vel $CP + CF = x + m$,

Ideoque $\overline{PF}^2 = mm \mp 2mx + xx$,

Sed $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$,

Hoc est, $tt \mp 2tz + zz = yy + mm \mp 2mx + xx$.

Sub-

Subſtituatur $\frac{mx}{t} = {}^{*}z$ in æquatione, loco ipſius z; & erit, * Art. 44.

$$tt \mp 2mx + \frac{mmxx}{tt} = yy + mm \mp 2mx + xx,$$

Ordinatis terminis prodit,

$$ttyy = t^4 - ttxx - mmtt + mmxx.$$

Unde $*\, yy : tt - xx :: tt - mm : tt$, * Art. 43.

Hoc eſt, $\overline{MP}^2 : AP \times Pa :: *\, \overline{BC}^2 : \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$. * Art. 42.

COROLLARIUM I.

47. QUADRATUM Ordinatæ cujuſcunque MK ad axem B*b*, eſt ad rectangulum ſub axis ipſius ſegmentis BK, K*b*, ut quadratum axis conjugati A*a*, ad quadratum ipſius B*b*.

Sit $Bb = 2c$. MK vel $CP = x$. CK vel $PM = y$.
Et erit,

$$*\, \overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2,$$ * Art. 46.

Hoc eſt, $yy : tt - xx :: 4cc : 4tt$,

Unde $*\, 4ttyy = 4cctt - 4ccxx$, * 16. El. 6.

Vel $4ccxx = 4cctt - 4ttyy$;

Ergo $*\, xx : cc - yy :: 4tt : 4cc$, * Art. 43.

Scilicet, $\overline{MK}^2 : BK \times Kb :: \overline{Aa}^2 : \overline{Bb}^2$.

COROLLARIUM II.

48. SIT Alteruter axium $Aa = 2t$. Ejus conjugatus $Bb = 2c$. Parameter autem ejus $= p$. Ordinatæ $PM = y$. Partes CP, inter centrum & Ordinatas, $= x$. His poſitis, * erit ſemper, Fig. 15. & 16. * Art. 46.

PM

$\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2.$

Vel $yy : tt - xx :: 4cc : 4tt.$

* Def. 8. Sed * $Aa : Bb :: Bb : p,$

* Cor. 2. 20. Elem. 6. Unde * $\overline{Aa}^2 : \overline{Bb}^2 :: Aa : p;$

Ergo $\overline{Bb}^2 : \overline{Aa}^2 :: p : Aa.$

Ideoque $yy : tt - xx :: p : 2t.$

Ductis autem in se invicem primæ & postremæ proportionis terminis extremis & mediis, erit $yy = cc - \frac{ccxx}{tt}$, Ita & $yy = \frac{1}{2}pt - \frac{pxx}{2t}$.

Et cum hæc proprietas de omnibus Ellipseos punctis vera sit, atque eorum situm quoad axes constanter definiat, sequitur æquationem $yy = cc - \frac{ccxx}{tt}$, ut & $yy = \frac{1}{2}pt - \frac{pxx}{2t}$, Ellipseos naturam quoad axes perfectè & ad amussim exprimere & determinare.

Corollarium III.

49. Hinc quadratum Ordinatæ cujusvis MP ad axem Aa, est ad rectangulum sub axis partibus AP, Pa, ut parameter p, ad ipsum axem Aa.

* Art. 48. Nam * $yy : tt - xx :: p : 2t,$

Hoc est $\overline{MP}^2 : AP \times Pa :: p : Aa.$

Cor-

COROLLARIUM IV.

50. SI ad axem Aa ducantur duæ Ordinatæ MP, NQ; quadrata earum erunt ad se invicem, ut rectangula $AP \times Pa$, & $AQ \times Qa$, sub axis partibus contenta.

Scilicet $\overline{MP}^2 : \overline{NQ}^2 :: AP \times Pa : AQ \times Qa$;

Nam * $\overline{MP}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$, * Art. 46.

Et $\overline{NQ}^2 : AQ \times Qa :: \overline{Bb}^2 : \overline{Aa}^2$,

Ergo $\overline{MP}^2 : AP \times Pa :: \overline{NQ}^2 : AQ \times Qa$,

Ideoque $\overline{MP}^2 : \overline{NQ}^2 :: AP \times Pa : AQ \times Qa$.

COROLLARIUM V.

51. SI per punctum quodvis P in alterutro axium conjugatorum Aa, agatur recta MM, alteri axi Bb parallela, ea Ellipsi in duobus punctis occurret a puncto P æquè remotis,

RECTAM MM Ellipsi occurrere manifestum est, ex eò quod axis Bb ei occurrat in punctis, B, b; ideoque *erunt MP, MP, ordinatæ; ac proinde earundem quadrata $\overline{MP}^2$, $\overline{MP}^2$, eidem quantitati $cc - \frac{ccxx}{tt}$ æquantur, ergo MP = MP. * Def. 7.

COROLLARIUM VI.

52. CUM sit * $yy = cc - \frac{ccxx}{tt}$, liquet, quo major fuerit CP, vel x, ex utravis parte centri C, hoc est, quò major * Art. 48.

jor fuerit quantitas subducenda $\frac{ccxx}{tt}$, eo minorem fore PM vel y; adeo ut, si CP vel x ipsi CA vel t fuerit æqualis, hoc est, si quantitas $\frac{ccxx}{tt}$ æqualis sit ipsi cc, prorsus evanescat PM vel y; & è contrario, quò minor fuerit CP vel x, eò major erit PM vel y; ideoque evanescente CP vel x, Ordinata PM (quæ in hoc casu fit CB) maxima erit omnium ad axem Aa Ordinatarum. Unde patet,

1*mo*. Si per terminos B, b, axis unius Bb, ducantur rectæ alteri Aa parallelæ, has esse tangentes in istis punctis B, b; & contra.

2*do*. Puncta Ellipseos inter A & B comprehensa, eo longius ab axe AP distare, quo remotiora sint a puncto A; illa autem, quæ inter B & a continentur, axi eò esse propiora, quo remotiora sint a puncto B.

Corollarium VII.

53. Si puncta P & P æquè remota sint a centro C, Ordinatæ MP, MP, ex eâdem axis parte sumptæ, erunt æquales; cadente enim puncto P supra centrum C, erit

* Art. 48. $\overline{PM}^2$ vel * $yy = cc - \frac{ccxx}{tt}$; erit etiam, puncto P infra C cadente, $\overline{PM}^2$ vel $yy = cc - \frac{ccxx}{tt}$; Sed cum CP vel x ex altera centri parte æqualis sit ipsi CP vel x ex altera, & cum cc & tt quantitates sint constantes, erit $cc - \frac{ccxx}{tt} = cc - \frac{ccxx}{tt}$, unde $yy = yy$; vel PM = PM. Ideoque si

si recta MM Ellipsi utrinque occurrens, ab uno conjugatorum axium Bb bifariam secetur in puncto K, ea erit alteri conjugato Aa parallela.

Ductis enim ad axem Bb parallelis MP, MP, constat rectam PP bifariam *secari in puncto C, quia MM bisecatur in K; ideoque PM, PM, erunt Ordinatæ æquales; unde *recta MM ipsi Aa parallela erit. *2 Elem. 6. *33. El. 1.

Corollarium VIII.

54. Si intelligamus Ellipseos partem BAb parti Bab superimponi, hæc illi perfectè congruet, hoc est, puncta A & M cadent super puncta a & M, cum omnes perpendiculares Aa, MM, bifariam secentur in punctis C & K; eodem modo probatur partem ABa ipsi Aba perfecte congruere & esse æqualem: unde liquet Ellipsin a duobus axibus secari in quatuor partes æquales, & similes, situ tamen diverso. Nam bAB & ABa singulæ semiellipsin conficiunt; auferatur communis pars ABC, & erit AbC = aBC. Simili ratiocinio constabit, partem ABC æqualem esse parti abC.

Propositio III.

Theorema.

55. *Si per terminum A ipsius Axis Aa, agatur recta LAL Conjugato Bb parallela, & in utrovis angulorum aAL, aAL, recta quæcunque AM; dico hanc rectam AM Ellipsi iterum occurrere in alio puncto M.*

Sumpta enim super AL, ex alterutra parte puncti A, recta AG ipsius axis Aa Parametro p æquali; agatur GF ipsi Aa parallela, rectæ AM (productæ, si opus) in Fig. 17.

 puncto

puncto F occurrens. Capiatur etiam super AL (ex eâdem axis parte, quà AM) pars AL ipsi GF æqualis, & ab altero axis Aa termino a ducatur aL; dico punctum M, ubi hæc recta ipsi AM occurrat, esse in Ellipsi MAM.

Sit enim MP ipsi AL parallela, & $Aa = 2t$. $AG = p$. GF vel $AL = a$. $CP = x$. $PM = y$. Triangula

* Art. 15. autem AGF, MPA, sunt * similia.

Unde AG : GF :: MP : PA,

Hoc est $p : a :: y : \frac{ay}{p}$ Ergo $AP = \frac{ay}{p}$.

* 2. El. 6. Porro ob triangula LAa, MPa * similia erit,

$AL : Aa :: PM : Pa$.

Scilicet $a : 2t :: y : \frac{2ty}{a}$, Unde $Pa = \frac{2ty}{a}$.

Erit igitur $AP \times Pa = \frac{2tyy}{p}$.

Sed $AP = t \mp x$. Et $Pa = t \pm x$, Prout punctum P supra vel infra centrum C cadit.

Ergo & $AP \times Pa = tt - xx$,

Ideoque $\frac{2tyy}{p} = tt - xx$,

Et $yy = \frac{1}{2}pt - \frac{pxx}{2t}$;

* Art. 48. * Erit igitur PM Ordinata ad axem Aa, hoc est, punctum M est ad Ellipsin.

Corollarium I.

56. Hinc ex dato Ellipseos axe Aa, parametro p, ductâque per istius axis verticem A recta AM, in utrovis angulorum aAL, aAL, facilè innotescit punctum M.

COROLLARIUM II.

57. MANIFESTUM est, nullam aliam rectam, præter LAL, Ellipsin tangere posse in puncto A; ea enim unica est, quæ per punctum A ducta, ipsam LL non secet; omnes igitur aliæ in angulo PAL necessariò cadent, ideoque Ellipsi occurrent in alio puncto M.

PROPOSITIO IV.

THEOREMA.

58. OMNES *Diametri* MC*m* *bifariam secantur in Centro* C, *& Ellipsi in duobus tantum punctis* M & *m* *occurrunt.* Fig. 17.

DUCTA Ordinatâ MP, sumatur C*p* ipsi CP æqualis; ex puncto autem *p* ducta *pm* ipsi C*p* normalis, diametro MC*m* occurrat in *m*; triangula igitur CPM, C*pm* sunt * similia, & † æqualia; unde CM = C*m*, & PM = *pm*. * 4. El. 6. † 26. El. 1.
Sed Ordinatæ a centro C æquidistantes sunt * æquales, * Art. 53.
& PM est ex Hypothesi Ordinata, ideoque *pm* erit etiam Ordinata, & punctum *m* est in Ellipsi.

PORRO si fingamus rectam aliquam ipsi B*b* parallelam a puncto C versus A moveri, liquet partem parallelæ intra angulum ACM contentam constanter augeri, partem verò inter rectam CM, & Ellipseōs portionem AM*b* inclusam, Ordinatam scilicet PM, continuo * diminui; unde patet rectam CM ad Ellipsin magis * Art. 52.
magisque accedere, usque dum ei occurrat in puncto M, postea autem magis magisque recedere; non igitur recta CM Ellipsi iterum occurrit in alio puncto præter M ex eâdem axis parte; & cum idem ostendi possit de recta

C*m*,

Cm, ſequitur Diametrum MCm Ellipſi in duobus tantum punctis M & m occurrere.

DEFINITIONES.

XII.

Fig. 18, 19, 20, 21. Si per Ellipſeos punctum quodvis M ducantur Diameter MCm, Ordinata MP ad alterutrum axem Aa, & recta quædam MT ita, ut ſit CT tertia proportionalis ad CP, CA; Diameter SCs ipſi MT parallela, dicitur eſſe Diametro MCm Conjugata; & viciſſim, Diameter Mm Conjugata eſt ipſi Ss, adeo ut Mm, Ss, ſimul ſumptæ appellentur Diametri Conjugatæ.

XIII.

Omnes rectæ ab Ellipſeos punctis ductæ, uni ex diametris conjugatis parallelæ, & ab altera terminatæ, vocantur Ordinatæ ad illam alteram; ita NO, Diametro Ss parallela, eſt Ordinata ad Conjugatam ejus Mm.

XIV.

Tertia proportionalis ad duas Diametros Conjugatas, vocatur Parameter primi in proportione termini; tertia ſcilicet proportionalis ad Mm, Ss, Parameter eſt Diametri Mm.

Corollarium.

* Def. 12. §. 9. Sit $CA = t$, $CP = x$, $PT = s$. Et * erit,

$$CP : CA :: CA : CT,$$

Hoc eſt, $x : t :: t : x + s$.

* 11. El. 6. Sed * $x : t :: t : \frac{tt}{x}$.

Ergo

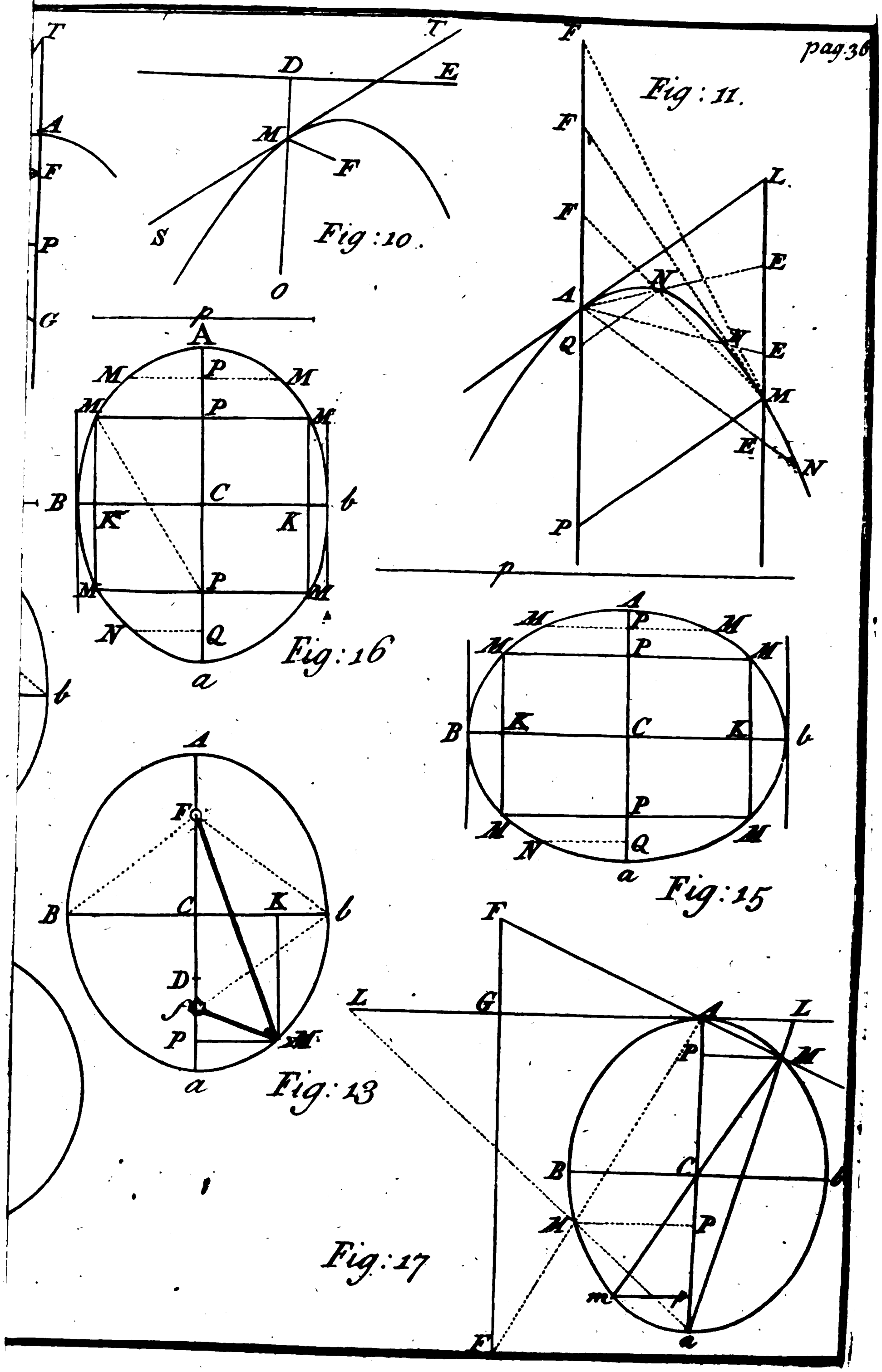


Fig: 10.

Fig: 11.

Fig: 16

Fig: 15

Fig: 13

Fig: 17

Ergo $x + s = \frac{tt}{x}$,

Vel $sx = tt - xx = AP \times Pa$.

PROPOSITIO V.

THEOREMA.

60. SI *per terminos* M, S, *duarum Diametrorum Conjugatarum* Mm, Ss, *agantur ad Axem* Aa *Ordinatæ* MP, SK; *dico Abscissam* CK *esse mediam proportionalem inter segmenta axis ab alterius Ordinatæ* MP *concursu facta*.

Scilicet * $\overline{CK}^2 = AP \times Pa$, * 17. El. 6.

Sit CA $= t$; CP $= x$; PT $= s$. CK $= m$. Et erit,

$AP \times Pa = tt - xx =$ * sx. * Art. 59.

Et $AK \times Ka = tt - mm$,

Sed * $tt = sx + xx$; * Art. 59.

Unde $AK \times Ka = sx + xx - mm$;

Hoc posito * erit, * Art. 50.

$$AP \times Pa : AK \times Ka :: \overline{PM}^2 : \overline{KS}^2;$$

Sed $\overline{PM}^2 : \overline{KS}^2 :: \overline{TP}^2 : \overline{CK}^2$ ob triangula TPM, CKS, * similia; * Art. 15.

Unde $AP \times Pa : AK \times Ka :: \overline{TP}^2 : \overline{CK}^2$,

Hoc est, $sx : sx + xx - mm :: ss : mm$.

Ergo * $mmsx = sssx + ssxx - mmss$, dividendo per s, & transponendo erit, * 16. El. 6.

$$mmx + mms = sxx + ssx;$$

Vel $mm = sx$,

Hoc est $\overline{CK}^2 = AP \times Pa$.

COR-

Corollarium.

61. Summa quadratorum ex duabus diametris conjugatis Mm, Ss, æquatur summæ quadratorum ex duobus axibus Aa, Bb.

$$\text{Dico } \overline{Mm}^2 + \overline{Ss}^2 = \overline{Aa}^2 + \overline{Bb}^2.$$

* 5. El. 2. Sit $CA = t$; $Bb = 2c$; $CP = x$. Et * erit

$\overline{CA}^2 - \overline{CK}^2 = AK \times Ka.$

* Art. 60. Est autem $\overline{CA}^2 = tt$; & * $\overline{CK}^2 = AP \times Pa = tt - xx.$

Unde $\overline{CA}^2 - \overline{CK}^2$, vel $AK \times Ka = xx.$

* Art. 48. Porro * $\overline{CA}^2 : \overline{CB}^2 :: AK \times Ka : \overline{SK}^2.$

Scilicet $tt : cc :: xx : \frac{ccxx}{tt} = \overline{SK}^2$;

Et $\overline{CA}^2 : \overline{CB}^2 :: AP \times Pa : \overline{PM}^2$,

Vel $tt : cc :: tt - xx : cc - \frac{ccxx}{tt} = \overline{PM}^2$,

* 47. El. 1. Jam vero * $\overline{CM}^2 = \overline{CP}^2 + \overline{PM}^2 = xx + cc - \frac{ccxx}{tt}$,

Et $\overline{CS}^2 = \overline{CK}^2 + \overline{KS}^2 = tt - xx + \frac{ccxx}{tt}$,

Unde $\overline{CM}^2 + \overline{CS}^2 = tt + cc$ (quantitatibus $+$ & $-$ se mutuo destruentibus,)

Sed $\overline{CA}^2 + \overline{CB}^2 = tt + cc$,

Ergo $\overline{CM}^2 + \overline{CS}^2 = \overline{CA}^2 + \overline{CB}^2$,

Hoc est $\overline{Mm}^2 + \overline{Ss}^2 = \overline{Aa}^2 + \overline{Bb}^2$,

PRO-

PROPOSITIO VI.

THEOREMA.

62. QUADRATUM *Ordinatæ cujusvis* ON *ad Diametrum* Mm, *est ad rectangulum* MO × Om *sub abscissis, ut quadratum ejus Conjugatæ* Ss, *ad quadratum ipsius Diametri* Mm;

$$\text{Dico } \overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2.$$

Ductis enim NQ, OH ad axem Bb, & OR ad ejus conjugatum Aa parallelis, occurrat OR Ordinatæ NQ (productæ, si opus) in puncto R.

Sit $CP = x$; $PM = y$; $CA = t$; $PT = s$; HQ vel $OR = a$. $CH = b$. Triangula autem CPM, CHO sunt * similia, *2. El. 6.

Unde CP : PM :: CH : HO,

$$\text{Vel } x : y :: b : \frac{by}{x} = HO \text{ vel } RQ.$$

Triangula etiam MPT, NRO sunt * similia, *Art. 15.

Unde TP : PM :: OR : RN.

$$\text{Vel } s : y :: a : \frac{ay}{s} = RN.$$

His positis erit in figuris 18, 19,

$$NQ = RQ - RN, \text{ vel } RN - RQ = \frac{by}{x} - \frac{ay}{s} \text{ vel } \frac{ay}{s} - \frac{by}{x},$$

$$\text{Unde } \overline{NQ}^2 = \frac{bbyy}{xx} - \frac{2abyy}{sx} + \frac{aayy}{ss}.$$

$$\text{Et in fig. 20, 21. } NQ = RQ + RN = \frac{by}{x} + \frac{ay}{s},$$

$$\text{Ergo } \overline{NQ}^2 = \frac{bbyy}{xx} + \frac{2abyy}{sx} + \frac{aayy}{ss}.$$

Ideo.

Ideoque in figuris 18, 19, 20, & 21, erit,

$$\overline{NQ}^2 = \frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{aayy}{ss}.$$

Rursus in Fig. 18, 19.

$$CQ = CH + HQ = b + a,$$

Et in Fig. 20, 21,

$$CQ = HQ - CH \text{ vel } CH - HQ = a - b \text{ vel } b - a,$$

ac proinde in figuris 18, 19, 20, 21,

Erit $\overline{CQ}^2 = aa \pm 2ab + bb$.

* Art. 50. Porro * $AP \times Pa : AQ \times Qa :: \overline{PM}^2 : \overline{QN}^2$,

Vel $tt - xx : tt - aa \mp 2ab - bb :: yy : \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx}$;

* 5. El. 2. (nam * $AQ \times Qa = \overline{CA}^2 - \overline{CQ}^2 = tt - aa - 2ab - bb$.)

Unde $\overline{NQ}^2 = \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx}$;

Sed $\overline{NQ}^2 = \frac{bbyy}{xx} \mp \frac{2abyy}{sx} - \frac{aayy}{ss}$,

Ergo $\frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{aayy}{ss} = \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx}$;

* Art. 50. Est autem $\mp \frac{2abyy}{xx} = \mp \frac{2abyy}{tt - xx}$, nam * $sx = tt - xx$.

Unde $\frac{bbyy}{xx} + \frac{aayy}{ss} = \frac{ttyy - aayy - bbyy}{tt - xx}$; dividendo per yy,

Erit $\frac{bb}{xx} + \frac{aa}{ss} = \frac{tt - aa - bb}{tt - xx}$.

Ductâ autem hâc postremâ æquatione in xx, & translatâ bb ad alteram partem, erit,

$$\frac{aaxx}{ss} \text{ vel } \frac{aax^4}{ssxx} = \frac{ttxx - aaxx - bbxx}{tt - xx} - bb.$$

Vel

Vel $\frac{aax^4}{ssxx} = \frac{ttxx - aaxx - bbtt}{tt - xx}$ (reductâ scilicet bb ad eandem denominationem cum fractione) unde (si ducatur primum æquationis membrum *viz.* $\frac{aax^4}{ssxx}$ in $ssxx$; alterum vero, in quadratum ipsius $tt - xx$, ipsi sx * æqualis; vel, quod eodem recidit, si numerator fractionis $\frac{ttxx - aaxx - bbtt}{tt - xx}$ in simplicem $tt - xx$ multiplicetur) erit, * Art. 59.

$$aax^4 = t^4xx - aattxx - bbt^4 - ttx^4 + aax^4 + bbttxx;$$

& auferendo utrinque aax^4, transponendo $- aattxx$, & dividendo per $ttxx$, evadit,

$$aa = tt - xx + bb - \frac{bbtt}{xx},$$

Hoc est, $\overline{OR}^2$ vel $\overline{HQ}^2 = tt - xx + bb - \frac{bbtt}{xx}$.

Jam vero si fiat CM vel C$m = z$; erit, ob triangula CPM, CHO * similia, * 2. El. 6.

$$CP : CM :: CH : CO,$$

$$\text{Vel } x : z :: b : \frac{bz}{x} = CO.$$

Unde MO $\times$ Om = * $\overline{CM}^2 - \overline{CO}^2 = \frac{xxzz - bbzz}{xx}$. * 5. El. 6.

Porro, triangula ORN, CKS, sunt * similia, * Art. 15.

$$\text{Unde } \overline{ON}^2 : \overline{CS}^2 :: \overline{OR}^2 : \overline{CK}^2,$$

Hoc est, $\overline{ON}^2 : \overline{CS}^2 :: tt - xx + bb - \frac{bbtt}{xx} :$ * $tt - xx$; * Art. 60.

Sed * $tt - xx + bb - \frac{bbtt}{xx} : tt - xx :: \frac{xxzz - bbzz}{xx} : zz$, * 16. El. 6.

$$\text{Unde } \overline{ON}^2 : \overline{CS}^2 :: MO \times Om : \overline{CM}^2,$$

Permutando $\overline{ON}^2 : MO \times Om :: \overline{CS}^2 : \overline{CM}^2 :: Ss : Mm$.

COROLLARIUM UNIVERSALE.

63 MANIFESTUM est quæcunque fuerint demonstrata in Propositione secundâ, de duobus axibus A*a*, B*b*; ope hujus Propositionis locum habere apud duas quasvis Diametros Conjugatas M*m*, S*s*; & cum articuli 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, a secundâ Propositione clarè deducantur, verique sint, utrum Angulus ACB rectus sit necne, constat, si ponamus rectas A*a*, B*b*, in istis Articulis non esse axes, sed duas diametros conjugatas quascunque, eos adhuc veros fore, demonstratio enim eorum prorsus eadem erit.

COROLLARIUM II.

Fig. 22. 64. CUM articuli 52 & 57, æque veri sint, sive A*a*, B*b* axes fuerint, sive Diametri conjugatæ quæcunque M*m*, S*s*; sequitur, lineam MT per terminum M Diametri cujusvis M*m* ductam, ipsi autem S*s* conjugatæ cum M*m* parallelam, esse tangentem in puncto M, eamque esse unicam, quæ Ellipsin in illo puncto tangere possit. Et vice versâ.

Unde constat, unam solummodo rectam Ellipsin in dato puncto tangere posse.

COROLLARIUM III.

65. HINC, si per Ellipseos punctum quodvis M, agatur ad alterutrum axem A*a* Ordinata quævis MP, & sumatur recta CT, ex parte puncti P, tertia Proportionalis ad CP, CA, & jungatur MT, liquet hanc rectam MT esse tangentem in puncto M; nam MT diametro S*s* est * parallela; ac proinde Ellipsin in puncto M * tangit.

* Def. 22. * Art. 64.

Et

Et vice versâ, si recta MT Ellipsin in puncto M tangat, & ducatur ad alterutrum axem Aa Ordinata MP; erunt partes CP, CA, CT continuè proportionales; ideoque $CP \times CT = \overline{CA}^2$.

COROLLARIUM IV.

66. SI in definitionibus 12, 13 & 14, & in duabus ultimis Propositionibus ponamus rectas Aa, Bb, non esse Axes, sed duas diametros quasvis conjugatas, constabit adhuc harum Propositionem veritas, eadem enim erit demonstratio, ut ex solâ Figuræ 22 inspectione patet, cum ex similibus triangulis CPM, CHO, & MPT, NRO, eadem eliciatur proportio, ac habitis Aa, Bb, pro Axibus Ellipseos.

Unde liquet consectarium præcedens æqualiter obtinere, sive Aa fuerit Axis, sive alia quævis Diameter;

Patet etiam, Diametros Conjugatas Mm, Ss, ex hac Hypothesi, pro axibus haberi posse; ideoque duos axes pro diametris habendos, quæ inter se rectos constituant angulos.

PROPOSITIO VII.

THEOREMA.

67. SI, *in Ellipsi ducantur diametri quævis conjugatæ* Mm, Ss, *axes autem* Aa, Bb; *dico parallelogrammum sub diametris* Mm, Ss, *æquari rectangulo sub axibus* Aa, Bb. Fig. 23.

Per puncta M, S, agantur rectæ MD, SD ipsis Ss, Mm parallelæ, sibi mutuo occurrentes in D; producta autem MD occurrat axi Aa producto in T; & a puncto M demittatur MP ipsi Aa normalis, & *erunt MD, SD tangentes in punctis M, S, & CMSD parallelogrammum sub semidiametris conjugatis CM, CS, & * æ- * Art. 52. 63. * 36. El. 1.

 quale

quale quartæ parti parallelogrammi circumscripti & sub ipsis diametris Mm, Ss, contenti, ex eo quod Mm, Ss
* Art. 58. *bifariam secentur in centro C. Demittatur CE ipsi DM
* Art. 47. normalis, & erit rectangulum $CS \times CE$ *æquale parallelogrammo CD; dico igitur esse $CS \times CE = CA \times CB = \frac{1}{4}$ rectanguli sub axibus; sit $CA = t$. $CB = c$. $CP = x$.

* Art. 46. Et erit *$\overline{CA}^2 : AP \times Pa :: \overline{CB}^2 : \overline{MP}^2$,

Hoc est $tt : tt - xx :: cc : cc - \frac{ccxx}{tt} = \overline{MP}^2$.

* 47. El. 1. Unde * $\overline{CM}^2 = xx + cc - \frac{ccxx}{tt}$.

* Art. 65. Porro * $CP : CA :: CA : CT$,

$$\text{Vel } x : t :: t : \frac{tt}{x}. \text{ Ergo } PT = CT - CP = \frac{tt}{x} - x,$$

$$\text{Et } \overline{PT}^2 = xx - 2tt + \frac{t^4}{xx}.$$

$$\text{Et } \overline{MT}^2 = xx + cc - 2tt + \frac{t^4}{xx} - \frac{ccxx}{tt}.$$

* 4 El. 6. Triangula autem MPT, CET sunt *similia ob angulos ad E & P rectos, & ob angulum ad T communem; unde

$$\overline{MT}^2 : \overline{MP}^2 :: \overline{CT}^2 : \overline{CE}^2,$$

$$\text{Vel } xx + cc - 2tt + \frac{t^4}{xx} - \frac{ccxx}{tt} : cc - \frac{ccxx}{tt} :: \frac{t^4}{xx} : \frac{t^4cc}{t^4 - ttxx + ccxx}$$

$$\text{Unde } \overline{CE}^2 = \frac{t^4cc}{t^4 - ttxx + ccxx}.$$

* Art. 61. Porro * $\overline{CS}^2 = \overline{CA}^2 + \overline{CB}^2 - \overline{CM}^2 = tt - xx + \frac{ccxx}{tt}$.

$$\text{Et } \overline{CE}^2 \times \overline{DM}^2 = \frac{t^8cc + t^4c^4xx - t^6ccxx}{t^6 + t^2c^2x^2 - t^4x^2} = ttcc,$$

Sed

Sed $ttcc = \overline{CA}^2 \times \overline{CB}^2$,

Ergo $\overline{CE}^2 \times \overline{CS}^2 = \overline{CA}^2 \times \overline{CB}^2$, nam $\overline{DM}^2 = \overline{CS}^2$;
Vel $CE \times CS$, hoc est parallelogrammum $CD = CA \times CB$.

COROLLARIUM.

68. HINC totum parallelogrammum circa diametros M*m*, S*s*, descriptum æquatur toti rectangulo sub axibus A*a*, B*b*; ideoque parallelogramma omnia, circa datæ Ellipseos diametros quasvis conjugatas descripta, erunt inter se æqualia; utpote quæ eidem rectangulo sub axibus facto æquentur.

PROPOSITIO VIII.
THEOREMA.

69. IN *Ellipsi, cujus centrum* C, *si per punctum quodvis* M *agatur ad axem* A*a ordinata* MP, *& ad tangentem* MT *per idem punctum ductam normalis quædam* MG; *dico esse semper* CP *ad* PG *in ratione data axis ipsius* A*a ad parametrum* p. Fig. 24.

Scilicet $CP : PG :: Aa : p$.

Sit CA vel $Ca = t$; $CP = x$; $PM = y$; & erit

$*CT = \frac{tt}{x}$, unde $PT = \frac{tt - xx}{x}$. * Art. 59.

Jam triangula TPM, MPG, sunt * similia, * 8. El. 6.

Ergo $TP : PM :: PM : PG$,

Vel $\frac{tt-xx}{x} : y :: y : \frac{xyy}{tt-xx} = PG$;

Unde $* \; x : \frac{xyy}{tt-xx} :: tt - xx : yy$, * 16. El. 6.

Hoc est, $CP : PG :: AP \times Pa : \overline{PM}^2$,

Sed

* Art. 49. Sed * $AP \times Pa : \overline{PM}^2 :: Aa : p$,
Ergo $CP : PG :: Aa : p$.

COROLLARIUM.

70. HINC rectangulum $TP \times PC : \overline{PM}^2 :: Aa : p$. nam, iisdem positis ac in propositione, * erit $TP \times PC = tt - xx$,

* Art. 49. Sed * $yy : tt - xx :: p : 2t$,
Vel $tt - xx : yy :: 2t : p$.

Hoc est $TP \times PC : \overline{PM}^2 :: Aa : p$.

PROPOSITIO IX.

THEOREMA.

Fig. 25, & 26. 71. SI *in Ellipsi ab extremitatibus* A, *a*, *Diametri cujusvis* A*a*, *agantur rectæ* AG, *ag*, *Ordinatis parallelæ*, *& alia quæpiam recta* MT *quomodocunque contingens ducatur, dico rectangulum sub rectis* AG, *ag*, *æquari quadrato dimidiæ Diametri* CB, *semiconjugatæ cum* A*a*.

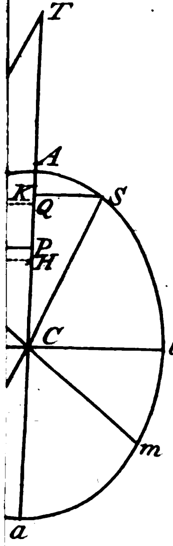
Scilicet $AG \times ag = \overline{CB}^2$.

Fig. 25. Si tangens Gg per extremitatem B transit, res constat; nam ob parallelas, $AG \times ag = \overline{CB}^2$.

Fig. 26. Non autem transeat per B; sed Gg, A*a*, productæ conveniant in T; ducanturque per punctum M recta MP ipsi AG, MO ipsi A*a* parallela;

* Art. 66. Quoniam * igitur $TC : CA :: CA : CP$,
* 19. El. 5. Erit * $TC : CA :: TA : AP$;
* Art. 58. Sed * $CA = Ca$,

Ergo



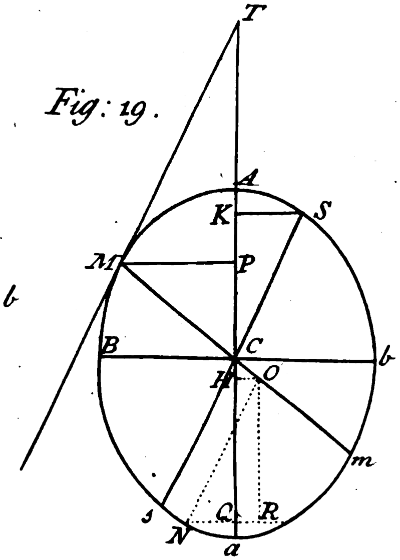


Fig: 19.

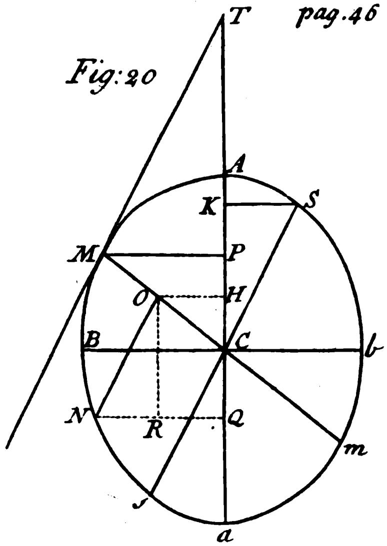
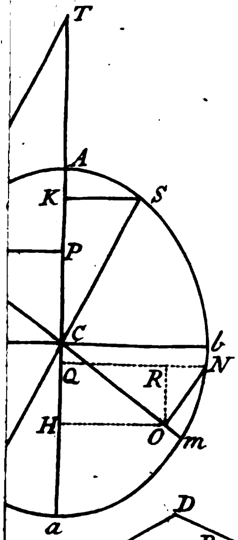


Fig: 20



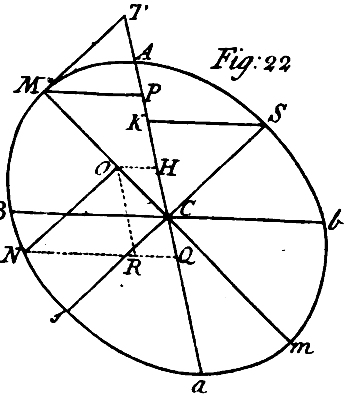


Fig: 22

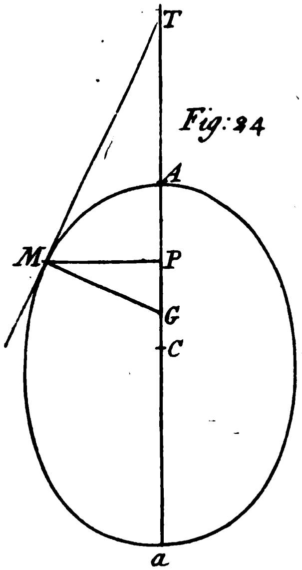


Fig: 24

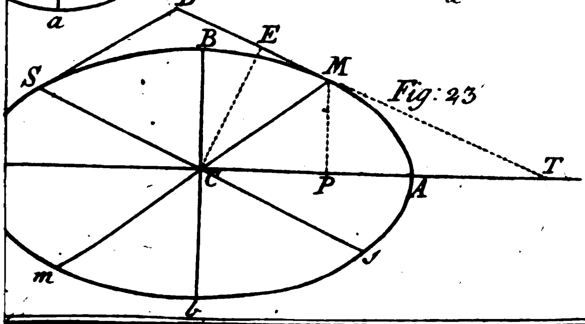


Fig: 23

Ergo $Ca : TC :: AP : AT$,

Et * $Ta : TC :: TP : AT$, *18. El. 5.

Unde * $ag : CN :: PM : AG$, *2 Elem. 6.

Et* $AG \times ag = PM \times CN$, *16. El. 6.

Sed $PM \times CN = CN \times CO = * \overline{CB}^2$, *Art. 65. & 66.

Ergo $AG \times ag = \overline{CB}^2$.

PROPOSITIO X.

THEOREMA.

72. SI *per Ellipseos punctum quodvis* M *agantur ad focos* F, *f, rectæ* MF, M*f*; *& per idem punctum tangens* TMS; *dico angulum* FMT *ab unâ tangentis parte* TM *& rectâ* MF *factum, æquari angulo* *f*MS *sub alterâ tangentis parte* MS *& rectâ* M*f* *comprehenso.* Fig. 27.

Scilicet angulum FMT $= f$MS,

Ductis enim ex punctis F, *f*, rectis FD, *fd*, tangenti TMS perpendicularibus, producatur axis A*a*, donec tangenti occurrat in T, & sit MP Ordinata ad axem.

Sit CA vel C$a = t$; CF vel C$f = m$; CP $= x$;

Et *erit $t - \frac{mx}{t} : t + \frac{mx}{t} :: \frac{tt}{x} - m : \frac{tt}{x} + m$, *16. El. 6.

Hoc est, *$MF : Mf :: TF$ vel $CT - CF : Tf$. * Art. 45.

Sed * $TF : Tf :: FD : fd$, *2. El. 6.

Unde $MF : Mf :: FD : fd$; ac proinde triangula FMD, *f*M*d*, erunt * similia; ideoque anguli FMD, *f*M*d*, vel FMT, *f*MS, lateribus homologis DF, *df* oppositi, erunt æquales. * 7. El. 6.

COR-

COROLLARIUM I.

73. HINC, si ab Ellipseos puncto quovis M ducantur ad focos F, f, rectæ MF, Mf; & per idem punctum M recta TMS, ita ut angulus FMT æqualis sit angulo fMS; constat rectam TMS Ellipsin tangere in puncto M.

COROLLARIUM II.

Fig. 28. 74. SI ab Ellipseos focis F, f, ad punctum quodvis tertium V inflectantur duæ rectæ FV, fV, quarum una fV axi majori Aa æqualis sit, altera FV a perpendiculo TMS in se demisso bisecetur in T, perpendiculum illud TMS Ellipsin tanget:

Secet enim perpendiculum TMS rectam fV (productam) in puncto M, & jungatur FM; ergo ob æquales TV, TF, & communem TM, erit latus
* 4. El. 6. *VM = MF, & angulus TMV = TMF. Unde erit MF
* Hyp. + Mf = MV + Mf = *Aa, axi majori: ideoque pun-
* Art. 39. ctum M * est in Ellipsi. Sed angulus TMF = TMV: &
* 15. El. 1. * fMS = TMV; unde TMF = fMS, & recta TMS
* Art. 73. Ellipsin * tangit in puncto M.

Et contra, si recta TMS Ellipsin tangit, erit fV æqualis Axi majori. Nam ob tangentem TMS erit an-
* Art. 72. gulus * FMT = fMS. Unde & FMT = TMV; sed anguli ad T sunt æquales, utpote recti. Unde triangula
* 26. El. 1. FMT, VMT * æquantur, ac proinde FM = MV. Sed
* Art. 39. FM + Mf = * Aa, axi majori; ergo MV + Mf = Vf = Aa.

PRO-

PROPOSITIO XI.

THEOREMA.

75. SI *per alterutrum terminum* A *Diametri* A*a ducatur recta* DAE, *conjugato ejus parallela, duabus aliis conjugatis diametris* M*m*, S*s*, *in punctis* D *&* E *occurrens; dico rectangulum sub istius parallelæ segmentis*, DA, AE, *æquari quadrato ipsius* BC, *dimidiæ Diametri* B*b*. Fig. 29.

Scilicet $DA \times AE = \overline{BC}^2$.

Ductis enim per extremitates M, S, diametrorum M*m*, S*s* ordinatis MP, SK, ad diametrum A*a*, fiat,

$CA = t$; $CB = c$; $CP = x$; $PM = y$;

Erit igitur * $\overline{CK}^2 = AP \times Pa = tt - xx$, * Art. 60.

Unde $\overline{CA}^2 - \overline{CK}^2 = tt - tt + xx = xx$,

Et * $AK \times Ka = \overline{CA}^2 - \overline{CK}^2 = xx$. * 5. El. 2.

Jam vero, * $\overline{BC}^2 : \overline{CA}^2 :: \overline{MP}^2 : AP \times Pa$, * Art. 48. & 63.

Hoc est $cc : tt :: yy : \frac{ttyy}{cc} = AP \times Pa$.

Ideoque $\overline{CK}^2 = \frac{ttyy}{cc}$; & $CK = \frac{ty}{c}$.

Porro * $\overline{CA}^2 : \overline{CB}^2 :: AK \times Ka : \overline{KS}^2$, * Art. 48. & 63.

Scilicet $tt : cc :: xx : \frac{ccxx}{tt}$. Unde $KS = \frac{cx}{t}$.

Ob * similia autem triangula CPM, CAD, erit * 2. El. 6.

$CP : PM :: CA : AD$,

Vel $x : y :: t : \frac{ty}{x} = AD$.

Ita & ob similia triangula CKS, CAE,

Erit CK : KS :: CA : AE,

Hoc est $\frac{ty}{c} : \frac{cx}{t} :: t : \frac{ccx}{ty}$. Unde AE $= \frac{ccx}{ty}$.

Ac proinde DA $\times$ AE $= \frac{ty}{x} \times \frac{ccx}{ty} = cc = \overline{BC}^2$.

COROLLARIUM.

Fig. 30. 76. HINC per articulum 66, si recta DME duobus axibus Aa, Bb, in punctis D & E occurrens, Ellipsin tangat; & per punctum contactus M agatur Diameter Mm; erit rectangulum sub tangentis partibus DM, ME, æquale quadrato semiconjugatæ CS cum Diametro Mm.

PROPOSITIO XII.

THEOREMA.

Fig. 30. 77. SI *tangat Ellipsin recta quælibet* ME, *axi in* E *occurrens, ac si a puncto contactus* M *agatur ad axem Ordinata* MP, *ut & è centro* C *recta* CS *tangenti* ME *parallela, æqualis vero semidiametro Conjugatæ cum diametro* Mm, *quæ per punctum contactûs* M *ducitur; dico quadratum tangentis* ME *esse ad quadratum semidiametri* SC *eidem parallelæ, sicut intercepta* EP *inter Ordinatam* MP *& punctum concursûs* E, *ad interceptam* PC *inter eandem Ordinatam* MP *& centrum* C.

Scilicet $\overline{ME}^2 : \overline{CS}^2 ::$ EP : PC.

Producta enim tangens EM, axi Bb, conjugato cum

* Art. 76. Aa, occurrat in D; & * erit DM $\times$ ME $= \overline{CS}^2$.

* 1. El. 6. Sed * $\overline{EM}^2 : \overline{DM} \times$ ME :: EM : MD.

* 2. El. 6. Unde $\overline{EM}^2 : \overline{CS}^2$ :: EM : MD :: * EP : PC.

LIBER TERTIUS DE HYPERBOLA

DEFINITIONES.

I.

SINT F & *f* puncta in plano aliquo fixa, & *f* MO Regula satis longa, cujus altera extremitas *f* puncto *f* ea lege applicetur, ut circa *f* tanquam centrum liberè rotari possit: jam si ad alteram Regulæ extremitatem O, & ad alterum punctum F, annectatur filum OMF, (quod regulâ vel longius vel brevius esse debet). & circumagatur Regula circa centrum *f*, interea dum filum OMF, ope styli vel paxilli M, continuo tendatur, & pars ejus MO Regulæ arctè juncta & velut agglutinata teneatur, curva AX paxilli motu descripta, appellatur HYPERBOLÆ PORTIO. Fig. 31.

CONVERSA Regula ad alteram partem puncti F, reliqua HYPERBOLÆ PORTIO simili prorsus motu describetur.

SED si, manente eâdem & Regulæ & fili longitudine, Regulæ extremitas puncto F, filum autem puncto *f* adjungatur; eadem ratione describi potest alia curva *xaz*, priori XAZ omnino similis & æqualis & ad verticem

ticem opposita, quæ etiam HYPERBOLA dicitur; adeo ut XAZ, *xaz*, simul appellentur HYPERBOLÆ OPPOSITÆ.

II.

PUNCTA fixa F, *f*, vocantur Hyperbolæ FOCI, vel UMBILICI.

III.

RECTA A*a*, quæ ad Hyperbolas oppositas terminatur, utrinque autem producta per focos transit, vocatur PRIMUS, vel PRINCIPALIS, vel TRANSVERSUS AXIS, vel LATUS TRANSVERSUM.

IV.

PUNCTUM C, ubi recta A*a* bifariam secatur, CENTRUM Hyperbolæ dicitur.

V.

SI per centrum C agatur ad axem A*a* perpendicularis indefinita B*b*; & a puncto A, tanquam centro, radio autem CF describatur circulus ipsi B*b* in punctis B & *b* occurrens, pars B*b* istiusce perpendicularis appellatur AXIS SECUNDUS.

VI.

AXES A*a*, B*b*, simul sumpti CONJUGATI dicuntur, adeo ut primus A*a* ipsi B*b* sit CONJUGATUS, & vicissim, secundus B*b* ipsi A*a*.

VII. RECTÆ

VII.

Rectæ MP, MK, ex Hyperbolarum oppositarum punctis, M, M, ductæ ad alterutrum Axium conjugatorum parallelæ, & ab altero terminatæ, dicuntur esse ad hunc alterum Ordinatæ, vel Ordinatim applicatæ, adeo ut MP sit Ordinata ad primum axem A*a*, & MK ad secundum Bb.

VIII.

Tertia proportionalis ad duos Axes vocatur Parameter primi in proportione termini: ideoque si fiat A*a* : B*b* :: B*b* ad tertiam *p*, erit tertia illa *p* Parameter ipsius A*a*.

IX.

Omnes rectæ per centrum C ductæ, appellantur Diametri; & quæ Hyperbolis Oppositis occurrunt, vocantur primæ Diametri; quæ autem non occurrunt, etsi in infinitum productæ, vocantur Secundæ Diametri.

X.

Recta, quæ Hyperbolæ in uno tantum puncto occurrit, utrinque autem producta extra cadit, dicitur esse Tangens in illo puncto.

XI.

Illa axis vel diametri cujusvis pars, inter verticem & Ordinatam comprehensa, vocatur Abscissa.

SCHO-

SCHOLIUM.

Fig. 32. 78. Necesse est filum, in prima definitione adhibitum, aut longius sit aut brevius quam filum fMO; si enim ei æqualis esset, linea paxilli motu descripta ejusmodi foret, ut omnia illius puncta a duobus focis f, F, æqualiter distarent, quoniam, ablata ex filo & Regula communi parte MO, residuæ MF, Mf, semper essent æquales; unde constat lineam hac ratione descriptam fore rectam indefinitam Bb, per centrum C ductam ipsi Ff normalem.

Corollarium I.

Fig. 31. 79. Liquet ex prima Definitione, si ex puncto quovis M in Hyperbolis oppositis agantur ad focos F, f, rectæ MF, Mf, earum differentiam semper esse eandem, Scilicet $Mf - MF = MF - Mf$.

Nam $Of - OMF = Mf - MF$,

Et $OF - OMf = MF - Mf$;

Sed $Of - OMF = OF - OMf$,

Ergo $Mf - MF = MF - Mf$.

Corollarium II.

80. Cadente puncto M in A, necesse est recta MF cadat super rectam AF, recta autem Mf super Af; ita etiam cadente M in a, cadet Mf super af, & MF super aF; unde

$Mf - MF = Af - AF$,

Et $MF - Mf = aF - af$,

* Art. 79. Sed * $Mf - MF = MF - Mf$;

Ergo

Ergo $Af - AF = aF - af$;

Sed $Af - AF = Ff - 2AF$,

Et $aF - af = fF - 2af$,

Ideoque $Ff - 2AF = fF - 2af$; & $AF = af$.

Hinc patet, *1mo.*

Distantiam focorum Ff bisecari in centro C,

Nam * $CA = Ca$; & $AF = af$, *Def. 4.

Unde $CA + AF = Ca + af$.

2do. Differentiam rectarum MF, Mf, primo axi Aa semper æquari,

Nam in Hyperbola XAZ, erit

$Mf - MF = Af - AF$;

Sed $Af - AF = Af - af = Aa$,

Unde $Mf - MF = Aa$.

Et in Hyperbola opposita *xaz*, erit,

$MF - Mf = aF - af$,

Sed $aF - af = aF - AF = Aa$.

Unde & $MF - Mf = Aa$.

COROLLARIUM III.

81. CONSTAT ex definitione quinta, primò secundum axem Bb bifariam secari in C, nam ob radios AB, Ab æquales, * erit, * 47. El. 1.

$\overline{BC}^2 + \overline{CA}^2 = \overline{bC}^2 + \overline{CA}^2$, unde ablato communi quadrato $\overline{CA}^2$, evadit $\overline{BC}^2 = \overline{Cb}^2$ & $BC = Cb$.

2do. Si super axem Bb sumatur pars CE ipsi CA, dimidio axis primi Aa, æqualis; & agatur Hypothenusa AE; secundum Axem Bb primo Aa majorem, æqualem, vel minorem esse, prout recta CF major, æqualis, vel minor fuerit ipsa Hypothenusa AE;

SIT

* Def. 5. SIT enim CF major quam AE, & *erit Ab major quam AE, & $\overline{AC}^2 + \overline{Cb}^2$ majora quam $\overline{AC}^2 + \overline{CE}^2$; ablato igitur communi $\overline{AC}^2$, erit $\overline{Cb}^2$ majus quam $\overline{CE}^2$; ideoque Cb vel Bb major quam AC vel Aa. Atque ita reliqui casus patent.

3*tio*. SI super axem Aa, ex utraque parte centri C, sumantur partes CF, Cf, singulæ Hypothenusæ AB æquales; puncta F, f, esse binos Hyperbolæ focos, nam
* Def. 5. radius AB distantiæ focali semper *æqualis est.

COROLLARIUM IV.

82. UNDE ex datis duobus axibus Aa, Bb, cognito etiam Aa esse primum, patet methodus Hyperbolas op-
* Art. 81. positas describendi. *Inventis enim super primum axem Aa focis F, f, ad punctum F annectatur fili OMF extremitas altera F, altera autem O ad extremitatem O Regulæ OMf, quæ circa focum f, tanquam centrum,
* Art. 78. rotari possit, & quæ filo adhibito vel *longior vel bre-
* Art. 80. vior sit excessu vel defectu, ipsi Aa *æquali. Descriptis igitur, ope hujus Regulæ & fili, duabus Hyperbolis oppositis XAZ, xaz, sicut in prima Definitione, constat eas habere pro axe primo rectam Aa, pro secundo autem rectam Bb, ideoque esse Hyperbolas quæsitas.

QUO longior fuerit Regula OMf, manente focorum intervallo, eo majores erunt Hyperbolarum oppositarum portiones; adeo ut pro libitu augeri vel diminui possint portiones illæ, si & Regula & filum æqualiter augeantur vel diminuantur.

SED si, & Regula & focis non mutatis, longius tantum filum adhibeatur, Hyperbola alterius speciei designabitur; & si adhuc paulo longius, adhuc alterius; donec tandem, filo Regulæ duplæ æquali existente, recta linea loco Hyperbolæ rursus describatur. MUTA-

MUTATA autem focorum distantia, & differentia inter fili & regulæ duplæ longitudinem pariter mutatâ, Hyperbolæ ejusdem quidem speciei describi possunt, sed quarum partes similes magnitudine different.

COROLLARIUM V.

83. QUADRATUM ipsius CB, dimidii axis secundi, æquatur rectangulo sub AF & Fa, partibus axis primi Aa, inter focum F & terminos ipsius Aa comprehensis,

Dico $\overline{CB}^2 = AF \times Fa$.

Sit CF vel * AB $= m$; CA vel C$a = t$; & * Def. 5.

Erit * $\overline{BC}^2 = \overline{AB}^2 - \overline{CA}^2 = mm - tt$, * 47. El. 1.

Sed $AF = CF - CA = m - t$; & $Fa = m + t$.

Unde $AF \times Fa = mm - tt = \overline{BC}^2$.

PROPOSITIO I.

THEOREMA.

84. SI *ad primum axem* Aa *agatur Ordinata* MP, *&* *super eundem axem productum sumatur pars* AD *ipsi* MF *æqualis* (*ex parte scilicet foci* F, *quando punctum* M *cadit in Hyperbola* XAZ, *&* *ex parte foci* f, *quando cadit in Hyperbola* xaz) *dico esse* CA : CF :: CP : CD. Fig. 33, & 34.

Sit CA vel C$a = t$; CF vel C$f = m$; CP $= x$; PM $= y$; CD $= z$. Et erit in primo casu (Fig. 33.)

AD vel $MF = CD - CA = z - t$.

aD sive * $Mf = CD + Ca = z + t$, * Art. 80.

$FP = CF - CP$, vel $CP - CF = m - x$, vel $x - m$, prout punctum P supra vel infra focum F cadat.

$Pf = CP + Cf = x + m$.

Porro in secundo casu (Fig. 34) erit,

AD vel $MF = CD + CA = z + t$.

aD sive $Mf = CD - Ca = z - t$.

$FP = CP + CF = x + m$.

$Pf = Cf - CP$, vel $CP - Cf = m - x$, vel $x - m$; prout punctum P infra vel supra focum f cadat.

Unde in utrisque casibus (Fig. 33 & 34.) erit $MF = CD \mp CA = z \mp t$; & $PF = x \mp m$.

* 47. El. 1. Sed * $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$.

Hoc est,

$$zz \mp 2tz + tt = yy + xx \mp 2mx + mm.$$

Porro in utrisque casibus,

Erit $Mf = z \pm t$; & $Pf = x \pm m$.

Sed $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$.

Unde $zz \pm 2tz + tt = yy + xx \pm 2mx + mm$.

Ideoque, quantitatibus in priori æquatione respectivè subductis ex quantitatibus in posteriori, evadit,

$$4tz = 4mx, \text{ \& } tz = mx.$$

* Art. 43. Unde * $t : m :: x : z$ vel $\frac{mx}{t}$.

Hoc est, $CA : CF :: CP : CD = \frac{mx}{t}$.

Corollarium.

Fig. 33. 85. Hinc si fiat CA vel $Ca = t$; CF vel $Cf = m$; $CP = x$; semper erit in primo casu, $MF = AD = CD - CA = \frac{mx}{t} - t$. Et $Mf = aD = CD + CA = \frac{mx}{t} + t$, puncto M cadente in Hyperbola XAZ.

In

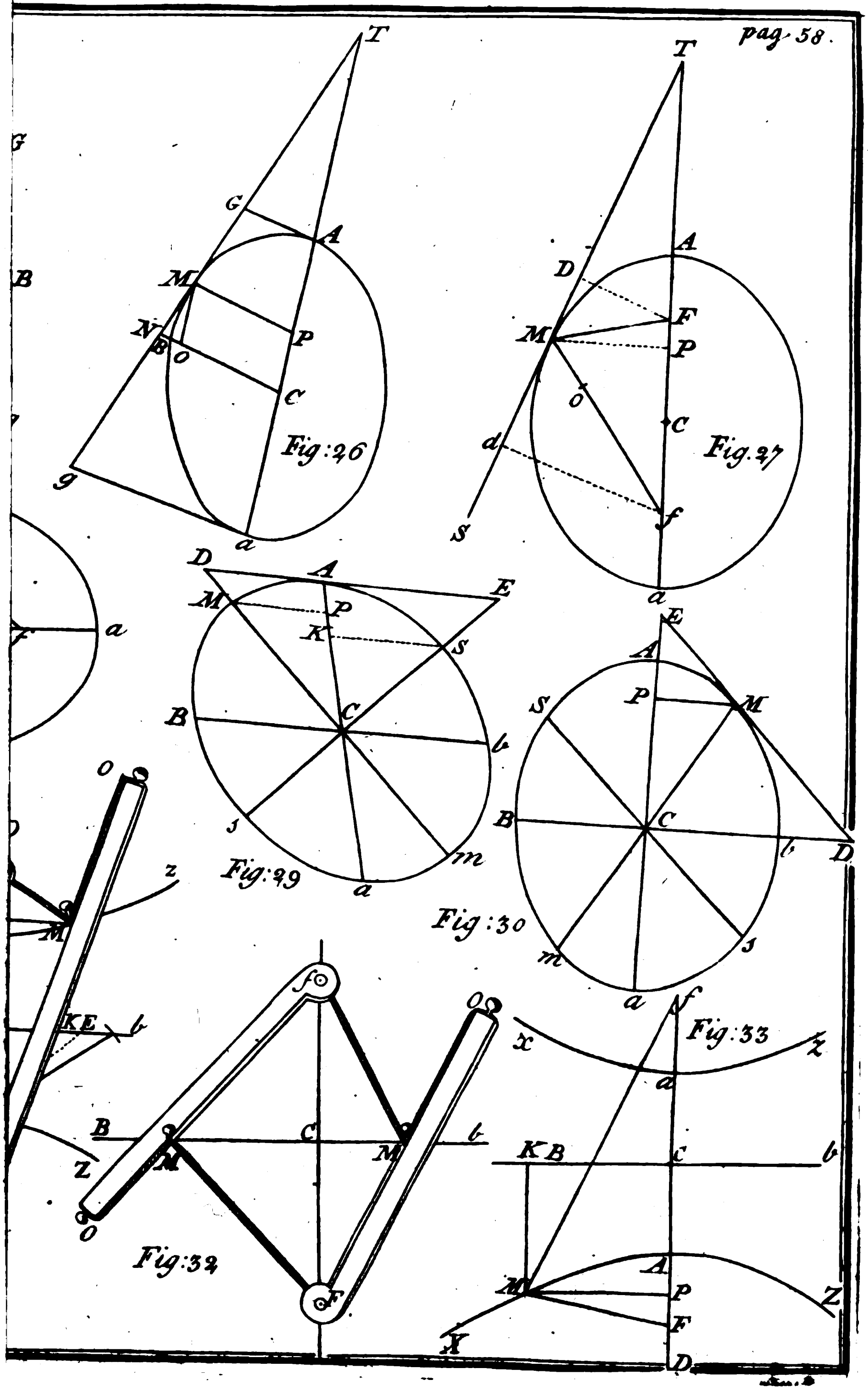


Fig: 26
Fig. 27
Fig: 29
Fig: 30
Fig: 32
Fig: 33

In secundo autem casu, puncto M cadente in Hyperbola *xaz*, erit MF $= \frac{mx}{t} + t$. Et M*f* $= \frac{mx}{t} - t$. Fig. 34.

PROPOSITIO II.

THEOREMA.

86. QUADRATUM *Ordinatæ cujusvis* MP *ad Axem* A*a*, *est ad rectangulum sub* AP *&* P*a* *illius axis producti partibus, ut quadratum conjugati* B*b* *ad ipsius* A*a* *quadratum.*

Scilicet $\overline{MP}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

Iisdem positis, quæ in præcedenti Propositione, ex triangulo rectangulo MPF eadem elicitur æquatio, ac prius,

Nempe $zz \mp 2tz + tt = yy + xx \mp 2mx + mm$.

Unde si in hac æquatione loco ipsius z substituatur $\frac{mx}{t}$ ipsi z * æqualis, oritur, * Art. 84.

$$yy + xx \mp 2mx + mm = \frac{mmxx}{tt} \mp 2mx + tt.$$

Unde $ttyy = mmxx - mmtt - tt + t^4$,

Ideoque * $yy : xx - tt :: mm - tt : tt$. * Art. 43.

Hoc est, $\overline{PM}^2 : AP \times Pa ::$ * $\overline{BC}^2 : \overline{CA}^2$. * Art. 83.

Sed $\overline{BC}^2 : \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$.

Ergo $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

COROLLARIUM I.

87. QUADRATUM Ordinatæ cujusvis MK ad secundum axem B*b*, est ad summam quadratorum ex CK,

 CB,

CB, ut quadratum conjugati Aa ad quadratum ipsius Bb.

Dico $\overline{MK}^2 : \overline{CK}^2 + \overline{CB}^2 :: \overline{Aa}^2 : \overline{Bb}^2$.

Sit $Bb = 2c$; est autem $MK = CP = x$; $CK = PM = y$.

* Art. 86. Jam verò * $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

Hoc est, $yy : xx - tt :: 4cc : 4tt$,

* 16. El. 6. Unde * $4ccxx = 4ttyy + 4cctt$,

* Art. 43. Ideoque * $xx : yy + cc :: 4tt : 4cc$.

Hoc est, $\overline{MK}^2 : \overline{CK}^2 + \overline{CB}^2 :: \overline{Aa}^2 : \overline{Bb}^2$.

Corollarium II et Fundamentale.

Fig. 35 & 36. 88. Sit Aa primus vel secundus axis $= 2t$, ejus conjugatus $= 2c$, Parameter $= p$; singulæ ordinatæ $PM = y$; partes CP, inter centrum & ordinatam, $= x$. His positis, erit semper

* Art. 86. * $yy : xx - tt :: 4cc : 4tt$,

Scilicet $\overline{PM}^2 : \overline{CP}^2 - \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$,

* Art. 87. Sed * $\overline{PM}^2 : \overline{CP}^2 + \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$,

Ergo $\overline{PM}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$.

Hoc est, $yy : xx \mp tt :: 4cc : 4tt$ (ubi observare est, signum esse $-$ in quantitate $xx \mp tt$, quando Aa est primus axis, signum vero esse $+$, quando Aa est secundus, ideoque loco ipsius $\overline{CP}^2 - \overline{CA}^2$ substitui potest rectangulum $AP \times Pa = xx - tt$.)

* Def. 8. Porro * $2t : 2c :: 2c : p$.

* 20. Elem. 6. Unde * $4tt : 4cc :: 2t : p$.

Ideoque $yy : xx \mp tt :: p : 2t$.

Ductis autem in ſe invicem extremis & mediis primæ & ultimæ proportionis terminis, evadit

$$yy = \frac{ccxx}{tt} \mp cc, \text{ ita \& } yy = \frac{pxx}{2t} \mp \tfrac{1}{2}pt.$$

Et cum hæc proprietas de omnibus Hyperbolarum oppoſitarum punctis vera ſit, atque eorum poſitionem quoad axes conſtanter definiat, ſequitur æquationem $yy = \frac{ccxx}{tt} \mp cc$, vel $yy = \frac{pxx}{2t} \mp \frac{1}{2}pt$ Hyperbolarum oppoſitarum naturam quoad earum axes perfectè & ad amuſſim exprimere & determinare.

COROLLARIUM III.

89. HINC quadratum Ordinatæ cujuſvis MP ad axem primum A*a*, eſt ad rectangulum AP × P*a* ſub axis producti partibus AP, P*a*, ſicut Parameter *p* ad ipſum axem A*a*.

Nam * $yy : xx - tt :: p : 2t$, * Art. 88.

Hoc eſt $\overline{MP}^2 : \overline{CP}^2 - \overline{CA}^2 :: p : Aa$.

Sed * $\overline{CP}^2 - \overline{CA}^2 = AP \times Pa$, ergo $\overline{MP}^2 : AP \times Pa :: p : Aa$. * Art. 88.

COROLLARIUM IV.

90. DUCTIS ad Axem A*a* duabus Ordinatis MP, NQ, erit

$$\overline{MP}^2 : \overline{NQ}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CQ}^2 \mp \overline{CA}^2.$$

Nam * $\overline{MP}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$, * Art. 88.

Et $\overline{NQ}^2 : \overline{CQ}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$,

Ergo

Ergo $\overline{MP}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{NQ}^2 : \overline{CQ}^2 \mp \overline{CA}^2$.

Notandum est hic etiam, rectangulum $AP \times Pa$ substitui posse loco ipsius $\overline{CP}^2 - \overline{CA}^2$; ita & $AQ \times Qa$ loco ipsius $\overline{CQ}^2 - \overline{CA}^2$; nam $AP \times Pa = xx - tt = \overline{CP}^2 - \overline{CA}^2$; atque eadem ratione erit $AQ \times Qa = \overline{CQ}^2 - \overline{CA}^2$.

Corollarium V.

91. Si per punctum quodvis P in alterutro axium Aa (producto si fuerit primus) agatur recta MPM Conjugato Bb parallela, ea uni vel binis Hyperbolis oppositis occurret in punctis M & M a puncto P æquè remotis.

Cum enim rectæ MP, MP, axi Bb sint parallelæ, erunt etiam Ordinatis ipsis * parallelæ, ac proinde Hyper-
* Def. 7. bolis occurrunt, unde & MP, MP, etiam ordinatæ; ideo-
* Art. 88. que quadrata $\overline{MP}^2$, $\overline{MP}^2$ eidem quantitati $\frac{ccxx}{tt} \mp cc$ * æqualia sunt; unde $MP = MP$.

Corollarium VI.

Fig. 35 & 36. 92. Quoniam $yy = \frac{ccxx}{tt} \mp cc$, constat quo major fuerit CP, vel x, ex alterutrâ parte centri C, eo majorem fieri PM vel y, ex alterutra parte axis Aa sumptam; ideoque auctâ in infinitum CP vel x, æqualiter augebitur PM vel y.

Et contra, quo minor fuerit CP vel x, eo minor erit PM vel y; adeo ut, si in (Fig. 35.) CP æqualis fuerit ipsi

ipsi CA vel $Ca = t$ (quando Aa est primus axis, ideoque * $yy = \frac{ccxx}{tt} - cc$) prorsus evanescat PM vel y. * Art. 88.

Evanescente autem CP vel x (quando Aa secundus axis fuerit) singula PM, quæ * ex hac Hypothesi fit Fig. 29. CB vel $Cb = c$, minor erit omnium ex utravis parte centri C ordinatarum.

Unde liquet primò, si (in Fig. 36) per terminos B, b, axis primi Bb agantur rectæ secundo axi parallelæ, eas esse in istis punctis B, b, tangentes.

2do. Hyperbolas oppositas ab axibus earum conjugatis magis magisque in infinitum recedere, eoque remotiora esse Hyperbolæ cujusvis puncta ab axe secundo, quo longius a primo axe distant; hoc tantum discrimine, quod axis primus binis Hyperbolis oppositis in uno puncto occurrat, productus autem intus cadat; quum tamen secundus, totus cadat intra binas Hyperbolas oppositas, neque iis unquam occurrit, etsi in infinitum productus.

COROLLARIUM VII.

93. SEQUITUR etiam, ex eò quod $yy = \frac{ccxx}{tt} \mp cc$, si puncta P & P æqualiter distant a centro C, ordinatas PM, PM, æquales esse: nam si CP æqualis sit ipsi CP, erit $x = x$, ideoque quantitas $\frac{ccxx}{tt} \mp cc$ eadem erit in utroque casu (nam quantitates cc & tt sunt constantes) unde yy ex utraque centri parte idem erit quadratum; idcirco PM = PM.

Unde constat, si recta MM ad Hyperbolam unam vel ad binas terminata, bifariam secetur ab axe quovis Bb

in

in puncto K a centro C diverso, hanc esse conjugato A*a* parallelam.

Actis enim axi B*b* parallelis MP, MP; recta PP
* 2 Elem. 6. *bisecabitur in C, quoniam MM bisecatur in K; ideoque Ordinatæ PM, PM erunt æquales: unde recta
* 33. El. 1. MM ipsi A*a* parallela * erit.

COROLLARIUM VIII.

94. SI plani Hyperbolici pars una super axem A*a* ita convertatur, ut in alteram plani partem cadat, hoc est, si (in Fig. 36,) ubi axis A*a* est secundus, omnes perpendiculares, MP, BC, *&c.* ponantur super omnes perpendiculares MP, *b*C, *&c.* hæ illis perfectè congruent, ideoque puncta B, M, *&c.* cadent super puncta *b*, M, *&c.* & tota Hyperbola MBM super totam Hyperbolam M*b*M, ex eo quod omnes illæ perpendiculares B*b*, MM,
* Art. 91. *&c.* bifariam * secentur in punctis C & P, *&c.*

Eadem ratione probatur portiones Hyperbolicas MAM, M*a*M esse etiam inter se æquales, quando A*a* (in Fig. 35) est primus Axis.

DEFINITIONES.

XII.

Fig. 37. ACTIS ex centro C duabus rectis indefinitis CG, C*g*, ad rectas AB, A*b* (per terminum A Axis primi A*a* ad terminos B, *b*, Axis secundi B*b* ductas) parallelis; hæ duæ rectæ vocantur ASYMPTOTI Hyperbolæ MAM; productæ autem in alteram partem centri C, dicuntur ASYMPTOTI Hyperbolæ M*a*M.

XIII. QUA-

XIII.

Quadratum partis CG vel C*g* inter centrum C & punctum G, ubi recta AB vel A*b* Asymptoto occurrat, comprehensæ, appellatur Dignitas Hyperbolæ MAM, vel oppositæ M*a*M.

Corollarium I.

95. Manifestum est angulum GC*g* ab Hyperbolæ Asymptotis factum (vel angulum BA*b* ipsi GC*g* * æqualem) recto minorem, æqualem, aut majorem esse, prout axis secundus B*b* axe primo A*a* minor, æqualis, aut major fuerit. * 27. El. 1.

Sit enim secundus axis B*b* primo axe A*a* minor, & erit latus CB minus latere CA, & angulus CAB * minor angulo CBA; sed angulus BCA est * rectus: unde CAB minor semirecto. Eodem prorsus ratiocinio ostenditur CA*b* semirecto esse minorem; ergo totus angulus BA*b* vel GC*g* recto minor est. Similiter etiam reliqui casus facillime demonstrantur. * 18. El. 1. * Def. 5.

Corollarium II.

96. Rectæ omnes CG, GA, GB, C*g*, *g*A, *gb*, sibi mutuo æquantur;

Nam GC ipsi A*b* parallela * est, unde * Def. 12.
Erit * BG : GC :: BA : A*b*, * 4. El. 6.
Sed * BA = A*b*; ergo & BG = GC. * Def. 5.
Porro * BG : GA :: BC : C*b*; * 2. El. 6.
Sed * BC = C*b*. Ergo & BG = GA. * Art. 81.
Et * cum GA = C*g*; res patet. * 34. El. 1.

COROLLARIUM III.

97. DIGNITAS Hyperbolæ æquatur quartæ parti summæ quadratorum ex duobus semiaxibus. Scilicet $\overline{CG}^2 = \frac{\overline{CA}^2 + \overline{CB}^2}{4}$.

*Art. 96. Sit $CA = t$; $CB = c$; $CG = m$; & erit * $AB = 2m$.

*47. El. 1. Sed * $\overline{AB}^2 = \overline{CA}^2 + \overline{CB}^2$,

Hoc est, $4mm = tt + cc$,

Unde $mm = \frac{tt + cc}{4}$, Vel $\overline{CG}^2 = \frac{\overline{CA}^2 + \overline{CB}^2}{4}$.

PROPOSITIO III.

THEOREMA.

Fig. 37. 98. SI *in alterutra Hyperbolarum oppositarum per punctum quodvis* M, *agatur ad axem* Aa *normalis* Rr, *ipsi* Aa *occurrens in* P; *dico rectangulum sub* RM *&* Mr *æquari quadrato ipsius* BC, *dimidii axis secundi* Bb.

Scilicet $RM \times Mr = \overline{BC}^2$.

Sit $CA = t$; $CB = c$; $CP = x$; $PM = y$; & erunt

*Art. 15. triangula ACB, CPr * similia; & ob eandem rationem ACb, CPR sunt similia;

Unde AC : CB vel Cb :: CP : Pr vel PR,

Vel $t : c :: x : \frac{cx}{t} = Pr$ vel PR;

Unde $RM = PR \pm PM = \frac{cx}{t} \pm y$;

$Mr = Pr \mp PM = \frac{cx}{t} \mp y$;

Ideo-

Ideoque RM $\times$ M$r = \frac{ccxx}{tt} - yy$.

Sed * $yy = \frac{ccxx}{tt} - cc$; unde si loco ipsius yy, in quan- *Art. 88.

titate $\frac{ccxx}{tt} - yy$, substituatur $\frac{ccxx}{tt} - cc$;

Erit RM $\times$ M$r = \frac{ccxx}{tt} - \frac{ccxx}{tt} + cc = cc = \overline{BC}^2$.

Corollarium I.

99. Hinc patet, omnia Hyperbolarum oppositarum puncta in angulis sub Asymptotis earum comprehensis contineri.

Nam PR vel P$r = \frac{cx}{t}$, & $\overline{PR}^2$ vel $\overline{Pr}^2 = \frac{ccxx}{tt}$; Sed

* $\overline{PM}^2 = \frac{ccxx}{tt} - cc$; hoc est, $\overline{PM}^2$ minus quam $\overline{PR}^2$ vel *Art. 88.

$\overline{Pr}^2$; unde & PM minor quam PR vel Pr; & punctum M non est adeo remotum ab axe producto, ac puncta R & r; & cum idem de omnibus Hyperbolarum & Asymptotôn punctis ostendi possit, liquet nulla Hyperbolarum oppositarum puncta in angulis KCr, RCk, cadere posse, sed omnia in angulis RCr, KCk, contineri.

Corollarium II.

100. Si in Hyperbola, vel in Hyperbolis oppositis, per duo quævis puncta M, N, agantur duæ rectæ Rr, Kk, primo axi Aa perpendiculares, & ab Asymptotis termi-

terminatæ, liquet rectangula RM × Mr & NK × Nk esse æqualia, utpote quæ eidem quadrato $\overline{BC}^2$ æquentur.
* 14. El. 6. Unde constat * esse RM : KN :: Nk : Mr.

PROPOSITIO IV.

THEOREMA.

101. SI *in Hyperbola, vel Hyperbolis oppositis per duo quævis puncta* M, N, *agantur duæ rectæ* Hh, Ll, *sibi invicem parallelæ, & Asymptotis occurrentes in punctis* H, h, L, l; *dico* HM × Mh = LN × Nl.

Ductis enim per puncta M, N, rectis Rr, Kk primo axi Aa perpendicularibus, constat triangula MRH,
* Art. 15. NKL esse * similia,

Unde RM : KN :: HM : LN;
Sed RM : KN :: Nk : Mr,
Ideoque HM : LN :: Nk : Mr.
* Art. 15. Porro triangula Mrh, Nkl sunt * similia,
Unde Nk : Mr :: Nl : Mh,
Ergo HM : LN :: Nl : Mh;
* 16. El. 6. Et * HM × Mh = LN × Nl.

COROLLARIUM I.

102. SI recta NL ipsi MH parallela, per centrum C transeat, hoc est, si MH fiat CE, patet puncta L & l, in unum coitura ad C; ac proinde rectangulum NL × Nl foret quadratum $\overline{EC}^2$; unde, si in alterutra Hyperbolarum oppositarum ex puncto quovis E agatur ad centrum C recta CE, & per punctum quodvis M recta MHh ipsi CE parallela, Asymptotis occurrens in punctis H &

H & h, liquet quadratum ipsius CE æquari rectangulo HM $\times$ Mh.

COROLLARIUM II.

103. SI in alterutra Hyperbolarum oppositarum per punctum quodvis N ducatur recta Ll, Asymptotis occurrens in punctis L, l; Hyperbolæ autem alterutri in puncto n; partes LN, ln, istiusce rectæ Ll inter Asymptotos & Hyperbolarum puncta comprehensæ, erunt æquales.

Sit enim $LN = a$; $Nn = b$; $ln = c$; & erit $Nl = Nn \mp ln = b \mp c$; unde etiam $LN \times Nl = ab \mp ac$.

Sed * $LN \times Nl = HM \times Mh$; * Art. 101.

Et * $HM \times Mh = Ln \times nl$, * Art. 102

Ergo $LN \times Nl = Ln \times nl$,

Hoc est $ab \mp ac = bc \mp ac$;

Unde $ab = bc$, & $a = c$, vel $LN = ln$.

COROLLARIUM III.

104. SI in consectario præcedenti ponamus rectam Nn ad Hyperbolas oppositas terminatam, per centrum C transire, hoc est, ipsam Nn fieri primam Diametrum DE; liquet bina puncta L, l, coitura in centro C; ideoque foret NL ipsa EC, & nl ipsa CD; sed NL semper est * æqualis ipsi nl; unde & EC ipsi CD; constat igitur primam quamvis Diametrum bisecari in centro C. * Art. 103.

COR-

Corollarium IV.

105. Si duæ rectæ Mm, Nn, sibi invicem parallelæ, & ad Hyperbolam vel Hyperbolas oppositas terminatæ, Asymptoto occurrant in punctis H, L; dico rectangula $MH \times Hm$ & $NL \times Ln$ esse æqualia.

Productis enim, si opus, rectis Mm, Nn, usque dum alteri Asymptoto in punctis h, l, occurrant; erit primo

* Art. 103. * $MH = mh$, unde $HM \times Mh = MH \times Hm$. Sed & $NL = nl$,

Ergo $LN \times Nl = NL \times Ln$;

* Art. 101. Sed * $HM \times Mh = LN \times Nl$,

Igitur $MH \times Hm = NL \times Ln$.

PROPOSITIO V.

Theorema.

Fig. 38. 106. Si *in Hyperbola vel Hyperbolis oppositis per bina quævis puncta* M, N, *agantur duæ rectæ* MH, NL, *sibi ipsis parallelæ & ab Asymptoto alterâ terminatæ; ita & ab iisdem punctis aliæ duæ rectæ* Mh, Nl, *parallelæ, & ab Asymptoto alterâ terminatæ; dico rectangulum* $HM \times Mh = LN \times Nl$.

Hæc propositio eâdem prorsus ratione demonstratur, quâ præcedens.

Corollarium I.

Fig. 39. 107. Si rectæ MH, Mh, ita & NL, Nl, duabus Asymptotis parallelæ fuerint, constat parallelogramma $MHCh$, $NLCl$ (ita & triangula CHM, CLN ipsorum

* 41. El. 1. * dimidia) sibi invicem æqua i,

Nam

Nam *$HM \times Mh = LN \times Nl$, *Art. 106.
Unde *$HM : LN :: Nl : Mh$; *16. El. 6.
Ergo, cum anguli ad C æquentur, *ideoque angulus $HMh = LNl$, *erit parallelogrammum $MHCh = NLCl$. * 34. El. 1. * 14. El. 6.

COROLLARIUM II.

108. IISDEM positis, quæ in præcedenti consectario, manifestum est, $CH \times HM$ æquari rectangulo $CL \times LN$; nam ex illa Hypothesi erit *$Mh = CH$; & $Nl = CL$; Sed cum rectangulum *$HM \times Mh = LN \times Nl$, erit quoque $CH \times HM = CL \times LN$, hoc est, si a duobus quibusvis punctis M, N, in Hyperbola vel Hyperbolis oppositis, agantur binæ rectæ, MH, NL, alteri Asymptoto parallelæ, & ab altera terminatæ, erit semper $CH \times HM = CL \times LN$; ac proinde *$CH : CL :: LN : HM$. * 34. El. 6. * Art. 106. * 16. El. 6.

COROLLARIUM III.

109. QUONIAM terminus A axis primi Aa punctum sit in Hyperbola; & recta AG Asymptoto CG occurrens in G, alteri Cg sit parallela sequitur *esse semper $CH \times HM = CG \times GA$, hoc est, *quadrato ipsius CG; vel, per Def. 13, dignitati Hyperbolæ. * Art. 108. * Art. 96.
SIT igitur $CG = m$; $CH = x$; $HM = y$ & erit semper $CH \times HM = \overline{CG}^2$; vel $xy = mm$. Et cum hæc proprietas omnibus Hyperbolorum oppositarum punctis æquè conveniat, atque eorum positionem quoad Asymptotos constanter definiat, liquet æquationem $xy = mm$ Hyperbolæ naturam quoad Asymptotos perfectè & ad amussim exprimere & determinare.

COR-

COROLLARIUM IV.

110. CONSTAT ex eo quod $MH = y = \frac{mm}{x}$, quo magis augeatur GH vel x, eo magis diminui HM vel y (aucto enim fractionis denominatore, tanto diminuitur ipsa fractio) adeo ut, recta CH in infinitum producta, prorsus evanescat HM vel y; unde patet, Hyperbolam & Asymptoton CH ad invicem magis magisque accedere, ita ut earum distantia quantitate omni data minor sit, nunquam tamen sibi ipsis occurrere, quia punctum occursus non nisi ex infinitâ Asymptoti productione habetur: nunquam autem in infinitum producitur recta.

Idem & de alterâ Asymptoto Cg intelligendum.

COROLLARIUM V.

111. RECTÆ sicut Aa, per centrum C, & intra angulum sub Asymptotis, ex parte Hyperbolarum, comprehensum ductæ, binis Hyperbolis oppositis in uno tantùm puncto A vel a occurrunt, productæ autem intra Hyperbolas cadunt.

Ob angulos enim GCA, gCA, atque his ad verticem oppositos, patet rectam Aa magis magisque ab Asymptotis recedere; Hyperbolæ autem oppositæ ad eas continuò * accedunt.

* Art. 110.

Eæ etiam rectæ, quæ, sicut Bb, intra angulos, qui sunt deinceps, cadunt, Hyperbolis oppositis nunquam occurrent, etsi in infinitum producantur; nulla enim Hyperbolarum oppositarum puncta intra eos angulos cadere * possunt.

* Art. 99.

Unde perspicuum est, * primas Diametros intra angu-

* Def. 9.

los

los ab Asymptotis factos contineri, secundas autem intra angulos, qui sunt deinceps.

Corollarium VI.

112. Si per punctum quodvis H in altera Asymptotôn CE, agatur recta HM alteri Ce parallela, ea Hyperbolæ in uno tantum puncto M occurret, distantia enim ejus ab Asymptoto Ce eadem semper manet, Hyperbola autem ad Ce continuo accedit. Fig. 40.

Corollarium VII.

113. Hinc, si in Hyperbola MN per punctum quodvis M ducantur rectæ indefinitæ MH, M*h*, ad Asymptotos CE, C*e* parallelæ, patet,

1mo. Omnia Hyperbolæ oppositæ puncta in angulo HM*h* contineri, omnia enim intra angulum ab Asymptotis factum continentur;

2do. Binas Hyperbolæ portiones intra angulos *h*MK, HM*k*, ex utraque parte anguli HM*h* cadere, neque ulla ejus puncta in angulo KM*k*, ipsi HM*h* verticali, inveniri posse.

3tio. Rectas omnes, sicut MF, in angulo HM*h* inclusas, & versus partem F productas, Hyperbolæ oppositæ occurrere in puncto N, & intra cadere; hæ enim rectæ ab ipsis MH, M*h* (ac proinde ab Asymptotis, quæ rectis MH, M*h* parallelæ sunt) magis magisque recedunt.

Quòd si ad alteram partem puncti M producantur rectæ, sicut MF, intra Hyperbolam MN cadent; neque ei in alio puncto præter M occurrent.

4to. Rectas, sicut E*e*, in angulis, qui ipsi HM*h* deinceps sunt, cadentes, occurrere Asymptotis Hyperbolæ

per M ductæ: ideoque cadentibus his rectis intra aliquam Hyperbolæ portionem, illæ necessario occurrent isti portioni in puncto aliquo N, necesse est enim Asymptoto extra hanc portionem posito alicubi occurrat.

COROLLARIUM VIII.

114. SI per Hyperbolæ punctum quodvis M ducatur recta Ff uni Asymptotôn in puncto F, Hyperbolæ autem oppositæ Asymptoto in puncto f occurrens, producaturque ad N, ita ut fN ipsi FM æqualis sit; dico punctum N esse in Hyperbola opposita; recta enim Ff in angulo HMh cadit, ideoque oppositæ Hyperbolæ in

* Art. 103. puncto aliquo occurrit: punctum igitur N * est punctum occursûs.

Porro, si per quodvis Hyperbolæ punctum M agatur recta Fe ad Asymptotos terminata; & super Ee capiatur pars eN ipsi EM æqualis, dico punctum N esse in Hyperbola.

Ducta enim recta MH uni Asymptoto Ce parallela, & ad alteram CH terminata, sumatur super hanc alteram pars CL ipsi HE æqualis, & sit LN ipsi HM parallela:

* Art. 112. constat igitur rectam LN Hyperbolæ * occurrere in pun-

* Art. 108. cto aliquo N; & * erit CL vel HE : HM :: CH vel EL : LN. Sed & punctum N in recta Ee est ejusmo-

* 2. El. 6. di, ut sit * eN = EM, ex eo quod sit EH = CL. Ergo punctum N est & in Hyperbola & in recta Ee.

PROPOSITIO VI.

THEOREMA.

Fig. 40. 115. SI *per quodvis Hyperbolæ punctum* M *agatur recta* MH *alteri Asymptoto* Ce *parallela, & ad alteram* CH *termi-*

CH terminata in puncto H; sumatur autem super hanc alteram pars HD ipsi CH aequalis, & a puncto D per M ducatur recta DMd Asymptoto Ce occurrens in d, dico rectam DMd Hyperbolam tangere in puncto M.

Non enim; sed si fieri potest Hyperbolae rursus occurrat in O, & * erit DM = O*d*. Sed ob similia triangula DC*d*, DHM, erit DH : HC :: DM : M*d*. Unde cum DH * aequalis sit ipsi HC, erit DM = M*d*, ideoque M*d* = O*d*, quod est absurdum: non igitur recta DM*d* Hyperbolae rursus in O, sed in unico puncto M occurrit. * Art. 95. * Hyp.

Corollarium I.

116. Hinc, si recta DM*d* Hyperbolam tangat in M, partes DM, M*d*, erunt aequales; & vice versâ, si partes DM, M*d* fuerint aequales, erit DM*d* tangens in puncto M.

Unde liquet unam solummodo rectam DM*d*, ad Asymptotos terminatam, Hyperbolam tangere posse in eodem puncto M.

Corollarium II.

117. Si per punctum M, ubi recta DM*d*, ad Asymptotos CL, C*l*, terminata, Hyperbolam tangit, ducatur prima Diameter MC*m*, Hyperbolae oppositae occurrens in puncto *m*, & per *m* agatur E*e* tangenti DM*d* parallela; dico hanc rectam esse tangentem in puncto *m*. Fig. 41.

Triangula enim CMD, C*m*E, sunt * aequalia, & similia (ob latus CM ipsi C*m* * aequale, & ob angulos ad C, ita & ad D & E aequales) unde DM = *m*E. Eo- * 26. El. 1. * Art. 104.

dem prorsus modo ostenditur $Md = me$; unde Ee bifariam secatur in m, quia * Dd bisecatur in. M; recta igitur Eme Hyperbolam * tangit in puncto m.

* Art. 116.
* Art. 116.

Unde constat tangentes Dd, Ee per terminos M, m, primæ Diametri cujusvis Mm ductas sibi invicem esse parallelas, & æquales, si modo ad Asymptotos terminatæ fuerint.

SCHOLIUM.

Fig. 40.

118. Ostensum est (Art. 110) eo minorem esse HM, quo major fuerit CH: adeo ut recta CH infinitè aucta, necesse sit HM infinitè diminuatur vel evanescat: quod si CH infinitè augeatur, HD ipsi CH æqualis infinitè etiam augebitur; unde MD, HD, (si modo sibi invicem non occurrant, nisi in infinitum productæ) pro parallelis haberi possunt, ideoque in se mutuo cadent, quia ex infinita productione puncta M & H in unum coeunt; hoc est, Asymptoto CE, & Hyperbola ipsa in infinitum productis, Asymptotos CE pro tangente habenda est, & ejus terminus infinitè distans pro puncto contactûs. Idem dicendum de Asymptoto Ce; unde constat binas Asymptotos pro rectis infinitis habendas esse, quæ Hyperbolas oppositas in earum terminis tangant.

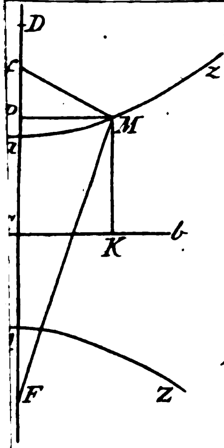
DEFINITIONES.

XIV.

Fig. 41.

Sint rectæ Mm, Ss, duæ Diametri, quarum una Ss tangentibus per terminos alteræ ductis sit parallela, & a rectis MS, Ms, ab extremitate M Diametri Mm ductis, & ad Asymptotos parallelis terminata; hæ duæ Diametri simul appellantur Conjugatæ.

XV. Rectæ



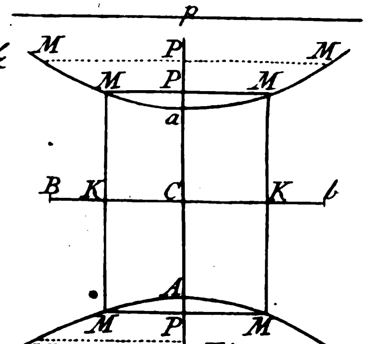


Fig: 35

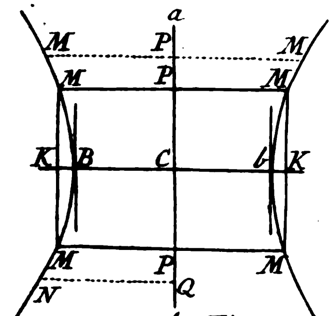
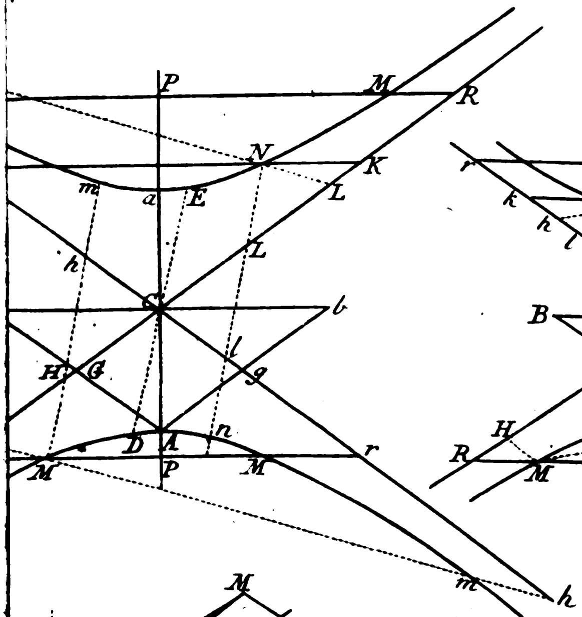


Fig: 36



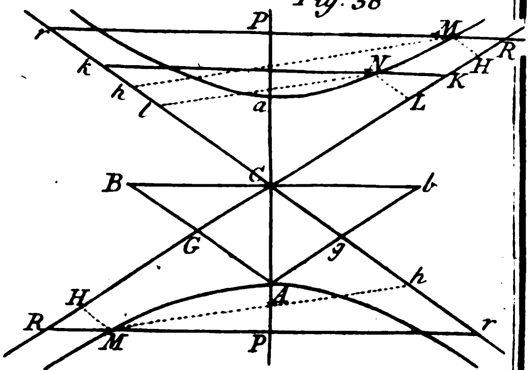


Fig: 38

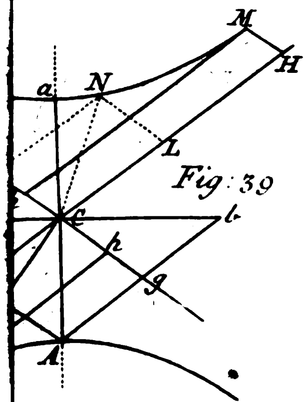


Fig: 39

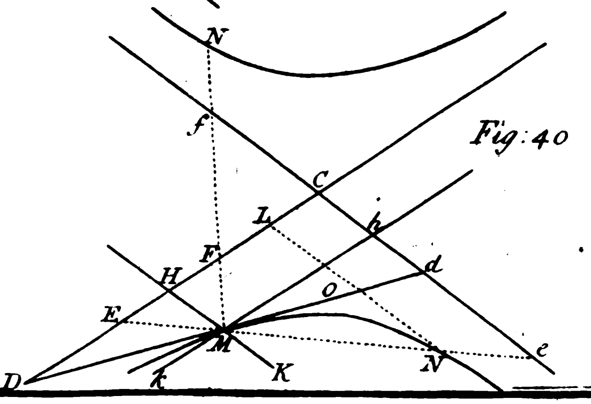


Fig: 40

XV.

RECTÆ omnes ab Hyperbolæ punctis ductæ ad unam ex Diametris Conjugatis parallelæ, & ab altera terminatæ, vocantur ORDINATÆ ad hanc alteram. Ita NO dicitur ORDINATA ad Diametrum M*m*.

XVI.

TERTIA proportionalis ad binas Diametros Conjugatas vocatur PARAMETER primi in proportione termini.

COROLLARIUM I.

119. DEFINITIO 14ª. duobus Axibus convenit; nam secundus Axis tangenti per extremitatem primi ductæ *parallelus est: & a duabus rectis ab una primi Axis extremitate ductis ad Asymptotos parallelis *terminatur. Unde constat duos Axes haberi posse pro duabus Diametris, quæ secum rectos constituant angulos. * Art. 92. * Def. 12.

COROLLARIUM II.

120. QUONIAM Diameter SC*s* tangenti DM*d* per terminum M Diametri MC*m* ductæ *parallela sit, & *cum hæc tangens binis Hyperbolæ per M ductæ Asymptotis CD, C*d*; occurrat in punctis D & *d*; sequitur hanc Diametrum SC*s* inter angulos cadere, qui sunt deinceps angulo DC*d* ab Asymptotis facto; ac proinde * esse secundam Diametrum. * Def. 14. * Art. 111.

Unde liquet ex duabus Diametris Conjugatis MC*m*, SC*s*, semper esse primam M*m*, & secundam S*s*.

COR-

COROLLARIUM III.

121. SECUNDA Diameter SC*s* bisecatur in centro C, & tangenti DM*d* per terminum M primæ Diametri M*m* ducte æqualis est.

* 34. El. 1. Ob parallelas enim MS, C*d*, & M*s*, CD, erit * MD = C*s*; & M*d* = CS. Unde tota DM*d* toti SC*s* æqua-
* Art. 116. lis: sed DM*d* * bisecatur in M, ergo & SC*s* in centro C.

COROLLARIUM IV.

122. DATIS duabus Diametris Conjugatis M*m*, S*s*, cognitâ etiam prima, facile inveniuntur Asymptoti CD, C*d*; agantur scilicet per centrum C rectæ CD, C*d*, ipsis MS, M*s*, per extremitatem M primæ Diametri M*m* ductis parallelæ, & erit factum.

Et vice versâ, datis duabus Hyperbolæ Asymptotis CD, C*d*, & puncto M, facile innotescunt binæ Diametri Conjugatæ MC*m*, SC*s*; agatur scilicet MH uni Asymptotôn C*d* parallela, alteri CD occurrens in H, & producatur MH, ita ut sit MH = HS, & jungantur CM, CS; fiat enim MD ipsi CS parallela, & erunt triangula CHS, MHD similia, ac proinde erit HM : HD
* Hyp. :: HS : HC. Sed * HM = HS, ergo & HD = HC.
* Art. 115. Erit igitur recta DM*d* * tangens in puncto M, & rectæ
* Def. 14. CM, CS * erunt semidiametri Conjugatæ.

COROLLARIUM V.

123. DATO secundæ Diametri SC*s* situ, facile innotescet ejus magnitudo, ita & prima Diameter MC*m* ipsi SC*s* conjugata.

Ducatur

Ducatur in angulo ab Asymptotis facto recta Ll ad Asymptotos terminata & ipsi SCs parallela: bisariam secetur Ll in puncto O, & agatur prima Diameter CO, Hyperbolæ occurrens in M; ductis igitur per punctum M rectis MS, Ms, ad Asymptotos Cd, CD, parallelis, patet puncta S, s, ubi hæ parallelæ Diametro SCs, occurrunt, magnitudinem ejus definire, & primam Diametrum MCm eidem esse conjugatam; ductâ enim per punctum M rectâ DMd ipsi Ll parallelâ, & ad Asymptotos terminatâ, ea bifariam * secabitur in puncto M, * 2. El. 6. quia Ll bisecatur in O, ideoque Dd * tangens est in * Art. 115. puncto M.

Unde liquido constat, si detur secundæ Diametri situs, ejus magnitudinem esse certam & definitam, unamque solummodo invenire posse; ita & primæ Diametri Mm, ipsi SCs conjugatæ, magnitudinem & situm determinari.

COROLLARIUM VI.

124. DATO secundæ Diametri SCs, & ejus Ordinatarum situ, data etiam magnitudine ejus, unà cum parametro, facile innotescunt situs & magnitudo primæ Diametri MCm ipsi SCs conjugatæ, una cum ipsius parametro.

Ductâ enim per centrum C indefinitâ rectâ Mm, Diametri Ss Ordinatis parallelâ, sumantur super rectam Mm duo puncta M & m a centro C æque remota, ita ut Mm sit media proportionalis inter secundam Diametrum Ss & parametrum ejus: inventâ deinde tertiâ proportionali ad binas rectas Mm, Ss, * constat Mm esse primam * Def. 15. Diametrum ipsi Ss conjugatam, & habere pro ejus pa- & 16. rametro istam tertiam proportionalem.

PRO-

PROPOSITIO VII.

THEOREMA.

Fig. 41. 125. QUADRATUM *Ordinatæ cujusvis* ON *ad Diametrum* Mm, *est ad rectangulum sub* MO *&* Om *illius Diametri segmentis, ut quadratum Conjugatæ* Ss *ad quadratum ipsius* Mm.

$$\text{Dico } \overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2.$$

Ductâ enim per extremitatem M Diametri primæ Mm rectâ Dd ad Asymptotos terminatâ, & secundæ Diame-
* Def. 14. tro Ss parallelâ, ea * erit tangens in puncto M, ideo-
* Art. 116. que * bisecabitur in M. Producatur igitur (ex utraque
* Def. 15. parte Diametri Mm) ordinata ON, ipsi Ss * parallela, & Asymptotis in punctis L & l occurret, a puncto O
* 2. El. 6. * æque remotis. His positis,
* Art. 121. Fiat CM vel $Cm = t$; CS vel Cs, sive *MD vel Md, $= c$; $CO = x$; $ON = y$.

Jam verò, triangula CMD, COL, sunt similia, Unde CM : MD :: CO : OL vel Ol,

Hoc est $t : c :: x : \frac{cx}{t} =$ OL vel Ol;

Idcirco $LN = LO \pm ON = \frac{cx}{t} \pm y$.

Et $Nl = lO \mp ON = \frac{cx}{t} \mp y$.

Unde $LN \times Nl = \frac{ccxx}{tt} - yy$,

* Art. 98. Sed * $LN \times Nl = DM \times Md$,

Vel $\frac{ccxx}{tt} - yy = cc$,

Ergo

Ergo $ccxx - cctt = ttyy$,

Et * $yy : xx - tt :: cc : tt$, * Art. 88.

Sive $\overline{ON}^2 : MO \times Om :: \overline{CS}^2 : \overline{CM}^2 :: \overline{Ss}^2 : \overline{Mm}^2$.

COROLLARIUM.

126. MANIFESTUM est, quæcunque in Propositione * secundâ de duobus Axibus A*a*, B*b*, fuerint demonstrata, ope hujus propositionis locum habere apud binas quasvis diametros conjugatas; & cum articuli 87, 88, 89, 90, 91, 92, 93, a secundâ Propositione clarè deducantur, verique sint, utrum angulus ACB rectus fuerit necne, sequitur, si in istis articulis ponamus rectas A*a*, B*b*, non esse axes, sed binas quasvis diametros conjugatas, eos adhuc veros fore, demonstratio enim eorum etiam ex hâc Hypothesi eadem prorsus erit. * Art. 86.

PROPOSITIO VIII.

THEOREMA.

127. SI *binæ rectæ* DE, FG *ad Asymptotos terminatæ, & se mutuo secantes in puncto* O, *Hyperbolam* MA *in punctis* M *&* A *tangant; dico triangulorum* CDE, CFG, *latera circa communem angulum* C *esse reciprocè proportionalia.* Fig. 42.

Scilicet CD : CF :: CG : CE.

Ductis enim per puncta contactuum, M, A, rectis MH, AL, Asymptoto CG parallelis, patet triangula CDE, HDM esse * similia, ac proinde CE : HM :: DE : DM; * 2. El. 6.
sed * DE = 2DM, unde CE = 2HM. Porro DE : EM * Art. 116.
:: DC : CH; unde etiam DC = 2CH.

Triangula autem CFG, LFA sunt similia; ergo CE $=$ 2CL, & CG $=$ 2AL, quia FG $=$ 2FA.

* Art. 108. Jam verò * 2CH : 2CL :: 2LA : 2HM,

Hoc est CD : CF :: CG : CE.

Corollarium I.

128. Hinc triangula ipsa CDE, CFG, communem

* 15. El. 6. angulum ad C habentia, erunt * æqualia.

Corollarium II.

129. Liquet rectam DE in eadem ratione secari ad puncta M & O, quâ FG ad puncta A & O.

* Art. 127. Nam * CD : CF :: CG : CE,

* 17. El. 6. Unde * erit DF : FC :: GE : EC; ideoque erit FE ipsi

* 2. El. 6. DG * parallela, & triangula FOE, DOG similia, unde,

Erit DO : OE :: GO : OF,

*18. El. 5. Et * DE : EO :: GF : FO,

*16. El. 5. Ergo * ME : MO :: AF : AO,

Hoc est, DM : MO :: GA : AO.

Lemma III.

130. Si *fuerint quæcunque & quotcunque magnitudines: ratio primæ ad ultimam componitur ex rationibus mediarum.*

Sint A, B, C, D, quantitates datæ, dico esse A ad D in ratione composita ex rationibus A ad B, B ad C, C ad D; nam ratio ipsius A ad D exprimitur per quotum ipsius A per D divisæ, hoc est, per $\frac{A}{D}$; atque idem de cæteris rationibus observandum; unde ratio composita

ex

ex intermediis rationibus æquatur (per Def. 5. Elem. 6.) exponentibus $\frac{A}{B}$, $\frac{B}{C}$ & $\frac{C}{D}$ in se mutuo ductis; videlicet $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{ABC}{BCD} = \frac{A}{D}$; Ergo ratio ipsius A ad D componitur ex rationibus mediis.

PROPOSITIO IX.

THEOREMA.

131. SI *per quodvis Hyperbolæ punctum* M *agatur ad diametrum quamvis* Aa *Ordinata* MP, *& tangens* MT *isti diametro occurrens in* T; *dico esse* CP : CA :: CA : CT; *necesse est autem ut puncta* P *&* T *ex eadem parte centri* C *cadant, quando* Aa *est prima Diameter*; *& è contra, ut* P *ex unâ, &* T *ex altera parte, quando* Aa *est secunda Diameter.* Fig. 43. & 44.

Si Aa fuerit prima diameter, producatur utrinque tangens MT, donec Asymptotis CD, CG, in punctis D & E occurrat; ita & Ordinata PM, donec Asymptoto CD in puncto N occurrat. Porro per punctum A agatur recta AK tangenti DE parallela, & Asymptoto CG occurrens in K; & sit FG tangens in puncto A, ad Asymptotos terminata, & alteri tangenti DE in O occurrens; & * erit FG ipsi MP parallela. * Def. 14. & 15.

His positis, erit AP ad AC in ratione compositâ ex EK ad EG, & ex EG ad EC.

Nam * AP : AC :: FN : FC, * 2. El. 6.

Et ratio ipsius FN ad FC, * componitur ex rationibus * Art. 130. FN ad FD, & FD ad FC.

Unde ratio AP ad AC composita est ex rationibus FN ad FD & FD ad FC.

* Art. 129. Sed FN : FD :: OM : OD :: * OA : OG :: EK : EG.

Ideoque ratio AP ad AC componitur ex rationibus EK ad EG, & FD ad FC.

Sed FD : FC :: EG : EC.

* Art. 127. (Nam * CD : CF :: CG : CE,

* 17. El. 5. Unde * FD : FC :: EG : EC.)

Ac proinde ratio AP ad AC componitur ex EK ad EG, & ex EG ad EC;

* Art. 130. Sed EK ad EC * componitur etiam ex rationibus EK ad EG, & ex EG ad EC.

Unde AP : AC :: EK : EC :: AT : TC.

Ergo AP : AC :: AT : TC,

* 18. El. 5. Ac proinde * CP : CA :: CA : CT. quod erat primum.

Fig. 44. *2do.* Jam sit Aa secunda Diameter; & agatur per centrum C recta CK Ordinatæ PM parallela, quæ Hyperbolæ in puncto B, tangenti autem MT in puncto R occurrat, & sit MK ipsi Aa parallela. Constat igitur CB

* Def. 14. * esse primam semidiametrum ipsi Aa conjugatam, &

* Def. 15. MK * esse ordinatam ad ipsam CB.

His positis, sit CA vel $Ca = t$; $CB = c$; CP vel $MK = x$; PM vel $CK = y$;

Et erit CK : CB :: CB : CR, per primam partem.

Hoc est, $y : c :: c : \frac{cc}{y} = CR$,

Unde $RK = CK - CR = \frac{yy - cc}{y}$;

* 4. El. 6. Triangula autem KRM, CRT sunt * similia,

Unde KR : RC :: MK : CT,

Vel

Vel $\frac{yy-cc}{y} : \frac{cc}{y} :: x : \frac{ccx}{yy-cc} = CT$,

Sed * $yy = \frac{ccxx}{tt} + cc$; ideoque $yy - cc = \frac{ccxx}{tt}$, unde si quantitas ccx dividatur per $\frac{ccxx}{tt}$ loco ipsius $yy - cc$, evadet, * Art. 88. & 126.

$$\frac{ccx}{yy-cc} = \frac{ccxtt}{ccxx} = \frac{tt}{x}; \text{ unde } CT = \frac{tt}{x};$$

Sed * $x : t :: t : \frac{tt}{x}$, vel CP : CA :: CA : CT, quod erat secundum. * 16. El. 6.

PROPOSITIO X.

THEOREMA.

132. SI *in Hyperbola, cujus centrum* C, *agatur ad alterutrum axem* Aa *Ordinata* MP; *ita & recta* MG, *tangenti* MT *per* M *ductæ perpendicularis; dico esse semper* CP *ad* PG *in ratione datâ ipsius axis* Aa *ad parametrum ejus* p. Fig. 45. & 46.

Scilicet CP : PG :: Aa : p.

Sit CA vel Ca = t; CP = x; PM = y; & erit * CT = $\frac{tt}{x}$; unde TP = CP ∓ CT = $x \mp \frac{tt}{x} = \frac{xx \mp tt}{x}$, prout axis A$a$ primus vel secundus fuerit. * Art. 131.

Jam vero triangula TPM, MPG sunt * similia, unde, * 8. El. 6.

TP : PM :: PM : PG,

Vel $\frac{xx \mp tt}{x} : y :: y : \frac{xyy}{xx \mp tt} = PG$,

Sed * $x : \frac{xyy}{xx \mp tt} :: xx \mp tt : yy$, * 16. El. 6.

Et

* Art. 88. Et * $xx \mp tt : yy :: 2t : p$,

Ergo, $x : \frac{xyy}{xx \mp tt} :: 2t : p$, hoc est CP : PG :: Aa : p.

COROLLARIUM.

133. Hinc si CA vel Ca dicatur t; CP, x; PM, y; Parameter, p; erit $\overline{PM}^2 : TP \times PC :: p : Aa$.

* Art. 132. Nam * $TP \times PC = xx \mp tt$;

* Art. 88. Sed * $yy : xx \mp tt :: p : 2t$; vel $\overline{PM}^2 : TP \times PC :: p : Aa$.

PROPOSITIO XI.

THEOREMA.

Fig. 47. 134. Si *in Hyperbola vel Hyperbolis oppositis, ab extremitatibus* A, a, *Diametri cujusvis* Aa *agantur rectæ* AG, ag, *Ordinatis parallelæ, & alia quæpiam recta* MT *quomodocunque contingens ducatur, dico rectangulum sub rectis* AG, ag, *æquari quadrato dimidiæ Diametri* CB, *semiconjugatæ cum* Aa.

Producta MG diametro Aa occurrat in puncto T; ducanturque per punctum M rectæ MP, MO, ipsis AG, Aa, parallelæ.

* Art. 131. Quoniam igitur * TC : CA :: CA : CP,

Erit TC : CA :: TA : AP; sed CA = Ca,

Ergo Ca : TC :: AP : AT,

Et Ta : TC :: TP : AT,

* 4. El. 6. Unde * ag : CN :: PM : AG,

Et AG × ag = PM × CN.

* Art. 131. Sed PM × CN = CN × CO = * $\overline{CB}^2$;

Ergo AG × ag = $\overline{CB}^2$.

PRO-

PROPOSITIO XII.

THEOREMA.

135. SI *ex quovis Hyperbolæ puncto* M *agantur ad focos* F, f, *rectæ* MF, Mf; *dico tangentem per punctum illud* M *ductam bifariam secare angulum* FMf. Fig. 48.

Scilicet $FMD = fMd$.

Ductis enim FD, fd tangenti MT perpendicularibus, sit Aa axis primus, ipsi MT occurrens in T, & sit MP ordinata ad axem.

Sit CA vel $Ca = t$; CF vel $Cf = m$; CP $= x$;

Et erit *$MF = \frac{mx}{t} - t$, & $Mf = \frac{mx}{t} + t$, * Art. 85.

Sed * $\frac{mx}{t} - t : \frac{mx}{t} + t :: m - \frac{tt}{x} : m + \frac{tt}{x}$, * 16 El. 6.

Hoc est, $MF : Mf :: CF -$ * $CT : Cf + CT$, * Art. 131.

Vel $MF : Mf :: TF : Tf :: FD : fd$,

Ergo $MF : Mf :: FD : fd$.

ideoque triangula FMD, fMd sunt * similia, ac proinde * 7. El. 6. anguli FMD, fMd, lateribus homologis DF, df, oppositi, sunt æquales.

COROLLARIUM I.

136. HINC si ab Hyperbolæ puncto quovis M, agantur ad focos F, f, rectæ MF, Mf, & per idem punctum M recta MT, ita ut angulus FMT æqualis sit angulo fMT, erit recta MT tangens in puncto M.

COR-

COROLLARIUM II.

Fig. 49. 137. Si ab Hyperbolæ focis F, f, ad punctum quodvis tertium V inflectantur duæ rectæ FV, fV, quarum una fV axi majori Aa æqualis sit, altera FV a perpendiculo MS in se demisso bisecetur in S; perpendiculum illud MS Hyperbolam tanget in M.

Secet enim perpendiculum MS rectam fV productam in puncto M, & jungatur FM. Et erit * VM = MF; & angulus SMV = SMF. Ergo Mf — MF = Mf — MV = Vf = Aa, axi majori; ideoque punctum M * est in Hyperbola; & cum angulus SMF = SMf, recta SM Hyperbolam tangit.

* 4. El. 1.

* Art. 80.

Et contra, si perpendiculum SM Hyperbolam tangit, erit fV axi majori Aa æqualis, nam ob æqualia triangula MSF, MSV, erit MF = MV, ideoque Mf — MV = Mf — MF = *Aa.

* Art. 80.

PROPOSITIO XIII.

THEOREMA.

Fig. 50. 138. DIFFERENTIA *quadratorum ex duabus Diametris quibusvis Conjugatis* Mm, Ss, *æquatur differentiæ quadratorum ex duobus axibus* Aa, Bb,

$$\text{Dico } \overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2,$$

$$\text{Vel } \overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2,$$

Actis enim rectis MS, AB, eæ Asymptoto alteri Cg erunt parallelæ, & ab altera CG bifariam sectæ in punctis H & G; nam ob parallelas Ms, Ch, erit SC : Cs :: SH : HM. Sed

Sed * $SC = Cs$; ergo & $SH = HM$. Eâdem ratione * ostenditur AB bisecari in puncto G; unde si ex punctis A, M, B, S, demittantur super Asymptoton CG normales AF, ML, BE, SK; triangula GAF, GBF, ita & HML, HSK, erunt similia, & * æqualia, ob latus $AG = GB$, & $SH = HM$. His positis, fiat CG vel * $GA = m$; GE vel $GF = a$; AF vel $BE = b$; $CH = x$; $HM = y$; & erit $CE = m + a$; $CF = m - a$.

* Art. 121.
* 26. El. 1.
* Art. 96.

Unde $\overline{CE}^2 + \overline{EB}^2 = \overline{CB}^2 = mm + 2am + aa + bb$,

Et $\overline{CF}^2 + \overline{FA}^2 = \overline{CA}^2 = mm - 2am + aa + bb$,

Ergo $\overline{CB}^2 - \overline{CA}^2 = 4am$.

Triangula autem GAF, HML sunt * similia, * Art. 15.

Ideoque $GA : AF :: HM : ML$,

Scilicet $m : b :: y : \frac{by}{m} = ML$ vel KS;

Sed & $GA : GF :: HM : HL$,

Hoc est, $m : a :: y : \frac{ay}{m} = HL$ vel HK,

ac proinde $CK = CH + HK = x + \frac{ay}{m}$,

Et $CL = CH - HL = x - \frac{ay}{m}$,

Ergo $\overline{CS}^2 = \overline{CK}^2 + \overline{KS}^2 = xx + \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$,

Et $\overline{CM}^2 = \overline{CL}^2 + \overline{LM}^2 = xx - \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$,

Unde $\overline{CS}^2 - \overline{CM}^2 = \frac{4axy}{m}$,

Sed * $xy = mm$; ac proinde substitutione factâ, erit * Art. 109. $\frac{4axy}{m} = 4am$, hoc est, $\overline{CS}^2 - \overline{CM}^2 = 4am$, sed & $\overline{CB}^2 - \overline{CA}^2 = 4am$, ideoque $\overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2$.

Si

Si angulus GCg ab Asymptotis factus esset acutus (etsi quidem in hac figura sit obtusus, atque ex hac hypothesi instituitur demonstratio) constat rectam CF majorem fore, quam CE; angulus enim AGE semper
* 34. El. 1. * æqualis est angulo GCg, & in hoc casu esset acutus; sed si perpendicularis AF inter puncta G & C caderet,
* 16. El. 1. angulus acutus AGE major * esset angulo recto, quod fieri nequit; unde recta AF infra G ex hac Hypothesi cadat necesse est; atque ita facilè ostenditur rectam BE supra G cadere, ex eo quod BGC tum esset acutus. Et in hoc casu eâdem prorsus ratione ostendi posset, esse

$\overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2$;

Sin vero angulus GCg ab Asymptotis factus fuerit rectus; AGE, isti angulo semper æqualis, esset quoque rectus; sed & ex hypothesi anguli CHM, CHS, essent recti; unde triangula CHM, CHS, ita & CGA, CGB, forent æqualia, & semidiameter CM = CS, ita & semiaxis CB = CA; & cum in hoc casu differentia inter binas diametros conjugatas Mm, Ss, ita & inter axes Aa, Bb, evanescat, liquido constat, hanc propositionem veram esse in omnibus casibus.

Corollarium.

139. Hinc patet, primam quamvis diametrum Mm secundâ Ss minorem, æqualem vel majorem esse, prout angulus ab Asymptotis factus fuerit obtusus, rectus, vel acutus; sit enim GCg obtusus, ideoque AGH & CHS, etiam obtusi; unde in triangulis CHS, CHM, angulus
* 24. El. 1. CHS major erit angulo CHM, ideoque * latus CS latere CM majus.

DEFINITIO. XVII.

Duæ Hyperbolæ oppositæ tum Æquilateræ, vel Rectangulæ dicuntur, cum binæ earum diametri conjugatæ sibi invicem æquales fuerint, vel cum angulus ab Asymptotis factus fuerit rectus.

Corollarium II.

140. Hinc, si ab Hyperbolæ Æquilateræ puncto quovis M agatur ad quamvis Diametrum Aa Ordinata MP, erit semper $\overline{MP}^2 = \overline{CP}^2 \mp \overline{CA}^2$; signo existente —, quando Aa est prima diameter; signo autem +, quando Aa est secunda; Fig. 51.

Nam * $\overline{MP}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$. Sed in Hyperbola Æquilatera $Bb = Aa$, ideoque $\overline{MP}^2 = \overline{CP}^2 \mp \overline{CA}^2$.

* Art. 88. & 126.

DEFINITIO. XVIII.

Sint AM, am, duæ Hyperbolæ oppositæ; recta Aa axis primus, recta autem Bb secundus: sint etiam BS, bs, duæ aliæ Hyperbolæ oppositæ, quarum è contrariò primus axis sit Bb, secundus Aa; hæ duæ Hyperbolæ BS, bs dicuntur esse alteris duabus AM, am, Conjugatæ; & hæ quatuor simul appellantur Hyperbolæ Conjugatæ. Fig. 52.

Corollarium.

141. Manifestum est, rectas B*a*, A*b*, sibi invicem esse parallelas, ex eo quòd rectæ A*a*, B*b*, bifariam * secentur in puncto C; unde constat, Hyperbolam BS ipsi AM conjugatam * habere pro altera Asymptoto rectam CG ipsius AM Asymptoton; & pro alterâ, rectam C*g* alteram Hyperbolæ AM Asymptoton versus C indefinitè productam, hæ enim rectæ per centrum C transeunt, & duabus rectis B*a*, BA, per terminum B, primi axis B*b* Hyperbolæ BS ad terminos A, *a*, axis secundi A*a* ductis sunt parallelæ.

* Def. 4. & 5.
* Def. 12.

Constat igitur rectas CG, C*g*, ipsis A*b*, AB, parallelas, & ex utrâque parte centri C indefinitè productas, non esse solummodo Asymptotos Hyperbolarum oppositarum AM, *am*, verum etiam binarum BS, *bs*, ipsis AM, *am*, Conjugatarum.

PROPOSITIO XIV.

Theorema.

142. Si *in alterâ Asymptotôn* CG *Hyperbolarum* AM, BS, *per punctum quodvis* H *agatur recta* MS, *alteri Asymptoto* C*g* *parallela; dico hanc rectam Hyperbolis* AM, BS, *occurrere in duobus punctis* M *&* S *a puncto* H *æquidistantibus.*

* Art. 112. Constat * enim 1*mo*, rectam MS Hyperbolis alicubi occurrere in punctis M & S.

* Art. 109. 2*do*. Ob Hyperbolam AM * erit rectangulum CH × HM = CG × GA; & ob Hyperbolam BS, erit CH × HS

* Art. 96. = GG × GB: sed * GA = GB; unde CG × GA = CG × GB; ideoque CH × HM = CH × HS, & HM = HS.

Cor-

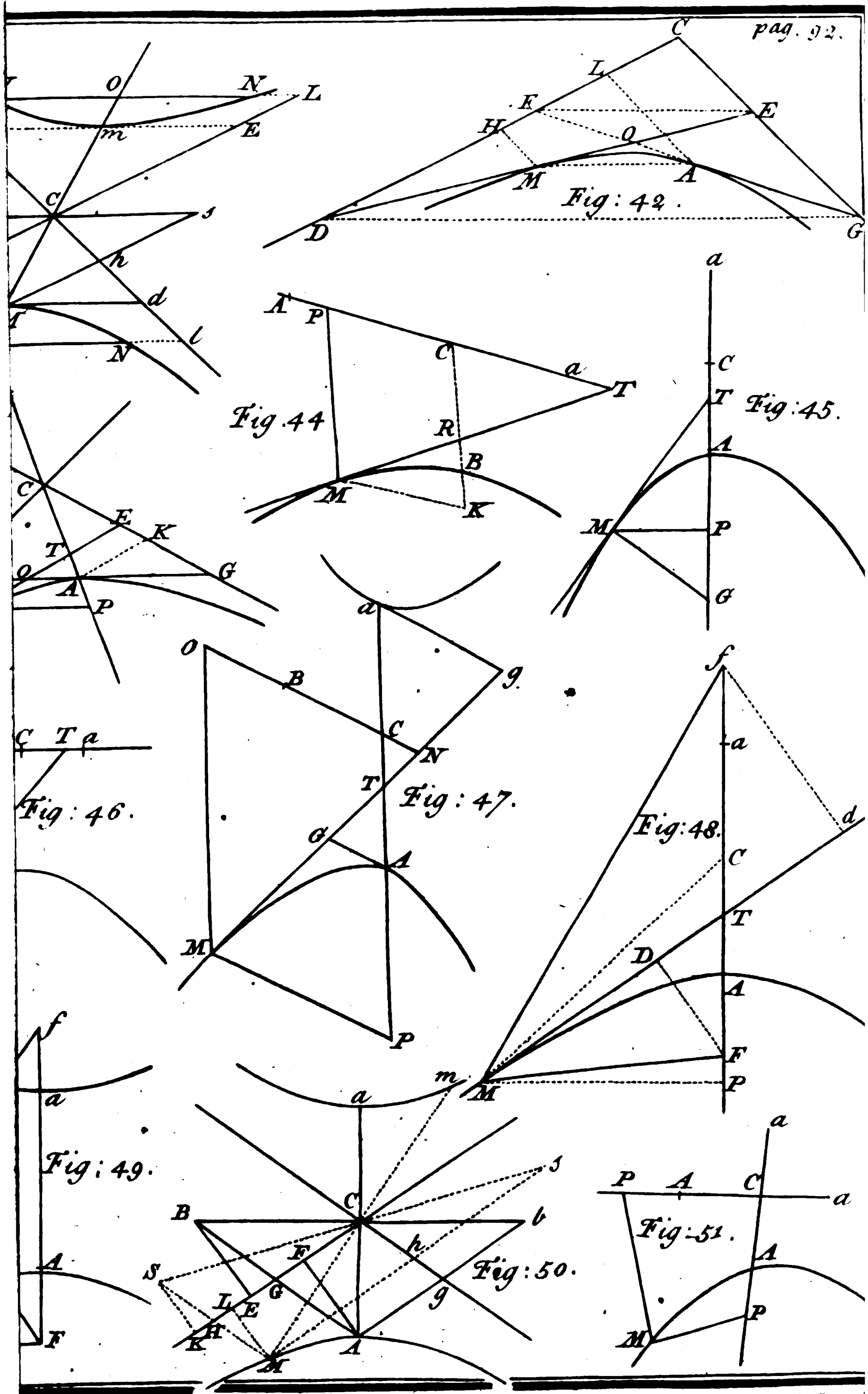


Fig: 42.
Fig. 44
Fig: 45.
Fig: 46.
Fig: 47.
Fig: 48.
Fig: 49.
Fig: 50.
Fig: 51.

COROLLARIUM I.

143. SI in duabus Hyperbolis AM, BS, per puncta M, S, ductæ intelligantur diametri duæ MC*m*, SC*s*, ad duas alias Hyperbolas *am*, *bs*, terminatæ, liquet in Hyperbolis oppositis AM, *am*, diametrum S*s* esse secundam, ac Conjugatam primæ M*m*; & vice versâ, diametrum M*m* esse in Hyperbolis BS, *bs*, secundam, primæ S*s* conjugatam; unde patet,

Binas quasvis diametros conjugatas in duabus Hyperbolis oppositis AM, *am*, esse etiam diametros conjugatas in duabus Hyperbolis BS, *bs*, ipsis AM, *am*, Conjugatis; hoc tantum discrimine, quòd prima diameter M*m*, quoad Hyperbolas BS, *bs*, sit secunda, & è contrariò secunda S*s*, prima.

COROLLARIUM II.

144. Unde liquidò apparet, Hyperbolas BS, *bs*, ipsis AM, *am*, Conjugatas, per terminos S, *s*, omnium diametrorum SC*s*, quoad Hyperbolas AM, *am*, secundarum transire; & vicissim Hyperbolas AM, *am*, ipsis BS, *bs*, Conjugatas per terminos M, *m*, diametrorum omnium MC*m*, quoad Hyperbolas BS, *bs*, secundarum, etiam transire.

PROPOSITIO XV.

THEOREMA.

145. SI *in Hyperbolis Conjugatis* AM, am, BS, bs, *ducantur diametri quævis Conjugatæ* Mm, Ss, *axes autem* Aa, Bb; *dico parallelogrammum sub diametris* Mm, Ss, *contentum, æquari rectangulo sub axibus* Aa, Bb. Fig. 55.

Per puncta M, S, agantur rectæ MD, SD, ipsis S*s*, M*m*, parallelæ, sibi mutuo occurrentes in D; producatur MD, donec axi A*a* occurrat in T; & a puncto M demittatur MP ipsi A*a* normalis; & *erunt MD, SD, tangentes in punctis M, S; & CMDS parallelogram- * Art. 92. & 126.

mum

mum, & æquale quartæ parti parallelogrammi circa diametros Mm, Ss, descripti, ex eo quòd Mm, Ss, bifariam secentur in centro C; demittatur CE ipsi MD productæ perpendicularis, & erit rectangulum $DM \times CE$ æquale parallelogrammo CD; dico igitur esse $DM \times CE$, vel $CS \times CE = CA \times CB$. Sit enim $CA = t$; $CP = x$; $CB = c$; & erit,

* Art. 86. $* \overline{CA}^2 : AP \times Pa :: \overline{CB}^2 : \overline{MP}^2$,

* Art. 88. Vel $tt : *xx - tt :: cc : \frac{ccxx}{tt} - cc = \overline{MP}^2$,

Unde $\overline{CM}^2 = xx + \frac{ccxx}{tt} - cc$.

* Art. 131. Porro $*\ CP : CA :: CA : CT$,

Vel $x : t :: t : \frac{tt}{x} = CT$,

Unde $PT = CP - CT = x - \frac{tt}{x}$, & $\overline{PT}^2 = xx - 2tt + \frac{t^4}{xx}$,

Ideoque $\overline{MT}^2 = xx - cc - 2tt + \frac{t^4}{xx} + \frac{ccxx}{tt}$.

Triangula autem MPT, CET, sunt similia,

Ergo $\overline{MT}^2 : \overline{MP}^2 :: \overline{CT}^2 : \overline{CE}^2$,

Vel $xx - cc - 2tt + \frac{t^4}{xx} + \frac{ccxx}{tt} : \frac{ccxx}{tt} - cc :: \frac{t^4}{xx} : \frac{-t^4cc}{t^4 - ttxx - ccxx} = \overline{CE}^2$.

* Art. 138. Porro $* \overline{CS}^2 = \overline{CM}^2 + \overline{CB}^2 - \overline{CA}^2 = xx + \frac{ccxx}{tt} - tt$;

Et $\overline{CE}^2 \times \overline{DM}^2$, vel $\overline{CE}^2 \times \overline{CS}^2 = \frac{t^8cc - t^6ccxx - t^4c^4xx}{t^6 - t^4xx - ttccxx} = ttcc$,

Unde $\overline{CE}^2 \times \overline{DM}^2 = \overline{CA}^2 \times \overline{CB}^2$, vel $CE \times DM = CA \times CB$.

COROLLARIUM.

146. HINC parallelogramma omnia circa datæ Hyperbolæ diametros quasvis conjugatas descripta, erunt inter se æqualia.

LIBER QUARTUS
DE TRIBUS
SECTIONIBUS CONICIS.

DEFINITIO.

SINGULA quæque Curva in præcedentibus libris descripta (videlicet PARABOLA, ELLIPSIS, HYPERBOLA, vel HYPERBOLÆ OPPOSITÆ) generaliter dici solet CONICA SECTIO.

PROPOSITIO I.

THEOREMA.

147. SI *in Ellipsi per terminum alterum* A *diametri cujusvis* Aa (*vel primæ diametri* Aa, *si sectio fuerit Hyperbola*) *agatur recta* AG *ordinatis parallela, æqualis autem parametro ad diametrum illam pertinenti; & a termino altero* a *ducatur recta* aG, *ordinatam quamvis* PM (*si opus, productam*) *in puncto* O *secans, dico quadratum Ordinatæ* PM *æquari rectangulo* AP × PO; Fig. 54 & 55.

Scilicet $\overline{PM}^2 = AP \times PO$.

Nam $Aa : AG :: Pa : PO :: AP \times Pa : AP \times PO$,

Sed * $Aa : AG :: AP \times Pa : \overline{PM}^2$, * Art. 49 & 89.

Unde $\overline{PM}^2 = AP \times PO$.

COR-

Corollarium.

148. Hinc quadratum Ordinatæ PM ad diametrum Aa (quod in Parabola (Fig. 56.) semper * æquale est rectangulo sub abscissâ AP & parametro AG) semper minus est in Ellipsi, majus vero in Hyperbola, quam prædictum rectangulum sub abscissa & parametro. Et ob has proprietates, Sectionibus Conicis nomina imposuit Apollonius, ita ut Parabola æqualitatem, Ellipsis defectum, Hyperbola autem excessum quadrati Ordinatæ denotaret, cum rectangulo sub Parametro & abscissâ comparati.

* Art. 7. & 23.

Propositio II.

Theorema.

Fig. 57. & 58.

149. In *Ellipsi omnis diameter* Aa, *& in Hyperbola omnis prima diameter* Aa, *bifariam secatur in centro* C, *neque Sectioni in pluribus, quàm duobus punctis occurrit.*

Hoc in articulis 58 libri secundi, 104 & 111 tertii demonstratur.

Propositio III.

Theorema.

150. Una *tantum tangens* LAL *transire potest per datum Conicæ Sectionis punctum* A.

Hoc in articulis 24 libri primi, 57 secundi, & 116 tertii ostenditur.

PRO-

PROPOSITIO IV.

THEOREMA.

151. SI *in Sectione Conicâ per punctum quodvis* P *diametri* Aa (*productæ in Hyperbola, quando* Aa *est prima diameter*) *agatur recta* MPM *istiusce diametri Ordinatis parallela; dico hanc sectioni occurrere in duobus punctis* M *&* M, *a puncto* P *æquidistantibus;* Fig. 59, 60, 61, 62.

Et vice versâ, si recta MM *ad Sectionem terminata bifariam secetur a diametro* Aa *in puncto* P *non centro, eâ istiusce diametri Ordinatis parallela erit.*

Hoc ostensum est in articulis 10, 12, 23, libri primi; 51, 53, 63, libri secundi; & 91, 93, & 126 libri tertii.

COROLLARIUM.

152. HINC si recta MM ad Sectionem Conicam terminata a diametro A*a* bifariam secetur in puncto P non centro, omnes rectæ ipsi MM parallelæ, & ad Sectionem terminatæ ab eadem diametro bisecabuntur.

PROPOSITIO V.

THEOREMA.

153. SI *duæ rectæ* MM, NN *sibi mutuo parallelæ, & ad Sectionem terminatæ, bifariam secentur in punctis* P *&* Q, *& per ea puncta ducatur recta* Aa; *dico hanc rectam esse diametrum.*

Diameter enim per medium punctum P rectæ MM ducta, *transibit etiam per medium Q ipsius NN. * Art. 152.

COROLLARIUM I.

* Def. 7. I. 154. DUCTA aliâ quâvis diametro D*d*, conſtat * Sectionem Conicam eſſe Parabolam, quando D*d* ipſi A*a* eſt parallela; * Ellipſin, quando D*d* ipſi A*a* intra Sectionem occurrit; Hyperbolam * vero, vel Hyperbolas oppoſitas, quando D*d* ipſi A*a* extra ſectionem in puncto C occurrit; ita tamen, ut in his duabus poſtremis caſibus punctum concursûs C ſit centrum.

* Def. 9. II.

* Def. 9. III.

Quando tota Ellipſis datur, centrum ejus facile innoteſcet, ductâ diametro A*a*, eâque bifariam ſectâ in puncto C. Eâdem prorſus ratione inveſtigatur centrum, datis Hyperbolis oppoſitis.

COROLLARIUM II.

155. HINC, datâ Conicâ Sectione, ita & puncto quovis O in eodem plano, ſemper duci poterit per illud punctum diameter D*d*; ſi enim Sectio ſit Parabola, agatur per datum punctum recta D*d* diametro cuivis A*a* parallela; ſi Ellipſis, vel Hyperbola, vel Hyperbolæ oppoſitæ, ducatur recta D*d* per punctum datum & per centrum C, ope conſectarii præcedentis inventum.

COROLLARIUM III.

156. HINC recta MM Sectioni Conicæ non niſi in duobus punctis M & M occurrere poteſt; ductâ enim per

* Art. 151. medium P ipſius MM diametro A*a*, * conſtat rectam MM iſtiuſce diametri Ordinatis eſſe parallelam, ideoque Sectioni in duobus tantum punctis M & M occurrit.

Si recta per centrum C tranſit, res patet per articulum 149.

COR-

COROLLARIUM IV.

157. DATA Ellipsi, vel Hyperbola, invenire binas ejusdem diametros conjugatas Aa, Bb; & Asymptotos CG, Cg, ducere, quando Sectio fuerit Hyperbola. Fig. 69. 70.

Inventâ, ope parallelarum MN, NN, diametro Aa, ductâque per centrum C rectâ Bb ipsis MM, NN, parallelâ, liquet * diametros Aa, Bb, esse conjugatas, ex eò quod rectæ MM, NN, a diametro Aa bifariam sectæ ad diametrum illam parallelæ erunt. * Def. 13. II. 15. III.

Jam verò ad ducendas Asymptotos CG, Cg, fiat $AP \times Pa : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ vel $\overline{Cb}^2$; vel (quod eodem recidit) ut media proportionalis inter AP & Pa ad PM, ita CA ad CB vel Cb. Ductis igitur rectis AB, Ab, agantur per centrum C rectæ indefinitæ Cg, CG, ipsis AB, Ab, parallelæ, & erunt Asymptoti quæsitæ; constat * enim Bb magnitudinem esse secundæ diametri Bb, ipsi Aa conjugatæ; unde, per definitiones 14 & 15 libri tertii, liquet propositum. * Art. 88. & 126.

PROPOSITIO VI.

PROBLEMA.

158. DATA *Sectione Conica, ita & ipsius diametro* Aa, *invenire istiusce diametri Ordinatarum situm.* Fig. 59, 60, 61, 62.

Actis ad diametrum datam Aa parallelis, sectioni in punctis M & M occurrentibus; dico rectam MM, quæ diametrum datam in puncto P secat, Ordinatam esse ex utrâque parte istiusce diametri, si modo punctum P non fuerit centrum.

Recta enim MM ex hypothesi bifariam secatur a di-
* Art. 151. ametro A*a* in puncto P, ideoque * erit Ordinata ex utraque parte diametri A*a*.

Hâc ratione semper licet invenire situm Ordinatæ
Fig. 59 & 61. PM ad diametrum A*a*; in Parabola enim, & in Hyperbola quando diameter data A*a* fuerit prima, liquet, quocunque intervallo rectæ diametro A*a* parallelæ a Sectione distant, eas Sectioni tandem occurrere in puncto aliquo M, ex eò quod Sectio magis magisque in infini-
* Art. 11. 23. 92 & 126. tum * recedat a diametro A*a*.

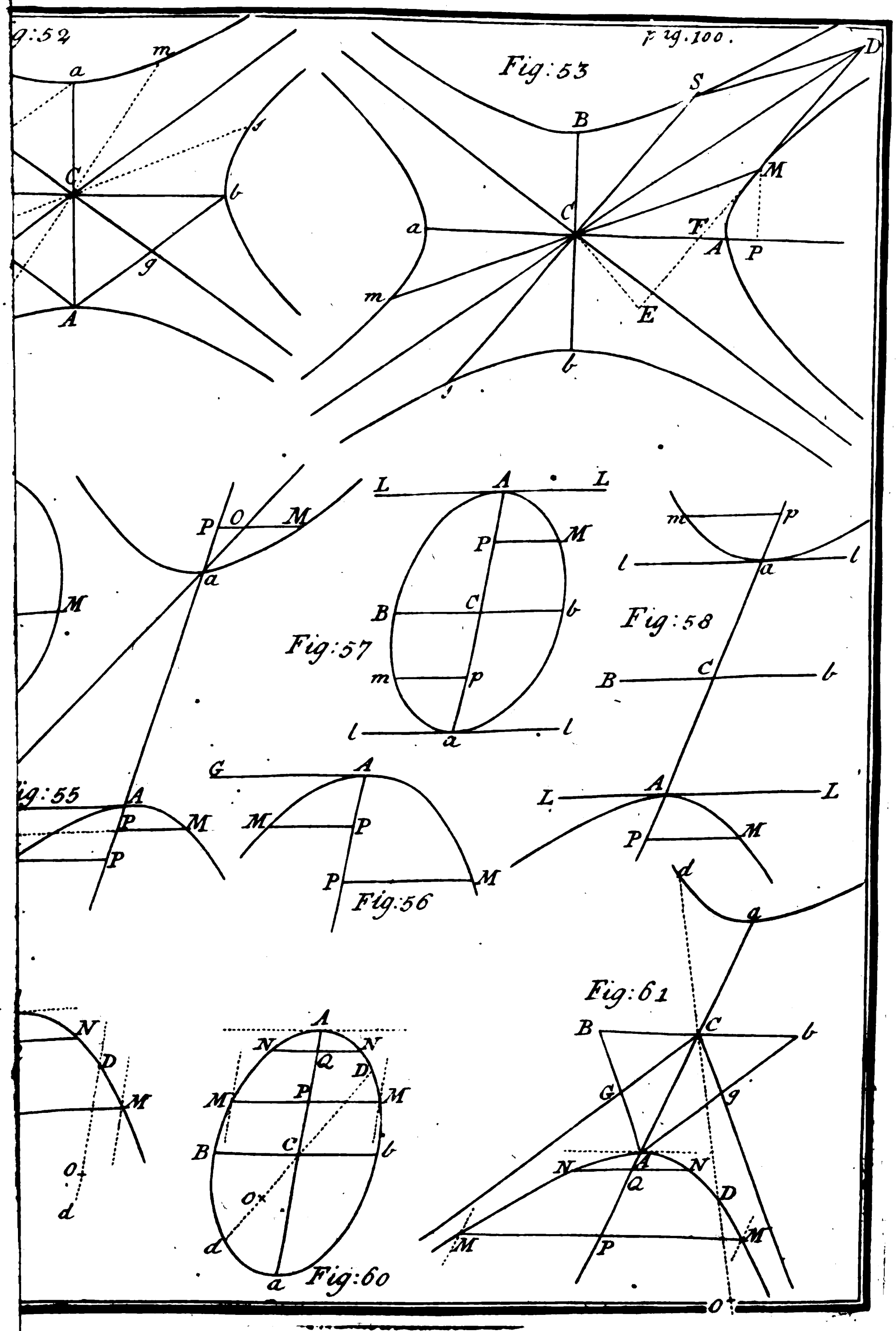
In Ellipsi (Fig. 60) & in Hyperbolis oppositis (Fig. 62) quando A*a* fuerit secunda diameter; liquet semper duci posse duas parallelas ex utraque parte diametri A*a*, sectionem in punctis M & M secantes, ita ut recta MM diametro datæ A*a* in puncto P, non centro, occurrat; ex eo, quòd in Ellipsi Ordinatæ ad diame-
* Art. 52. & 63. trum * A*a* eò minores, & è contrariò in Hyperbola eò
* Art. 92. 126. majores * sunt, quo remotiores a centro C.

Corollarium I.

159. Hinc duci potest tangens per punctum A Sectio-
* Art. 155. nis Conicæ datum, ducta * enim per illud punctum diametro A*a*, inventâque ad istam diametrum Ordinatâ
* Art. 11. 23. 52. 63. 92 & 126. Def. 10, I. 13, II. 7. III. MPM, * liquet, si per punctum A agatur ad rectam MM parallela, eam esse tangentem in A.

Corollarium II.

160. Datis Hyperbolis oppositis, vel Ellipsi, ita & ipsarum diametro quâvis A*a*, facile inveniri potest diameter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meter Bb ipsi Aa conjugata. Ducatur scilicet per centrum C recta Bb Ordinatis ad illa diametrum parallela.

Sit autem Bb diameter data, cujus conjugata Aa sit invenienda. Ducta MM ipsi Bb parallela, & ad sectionem terminatâ, bisecentur rectæ MM, Bb, in punctis P & C, & per P & C agatur Aa, & ea erit diameter quæsita.

COROLLARIUM III.

161. DATA Hyperbolâ MAM (Fig. 61.) ita & secundæ diametri Bb positione, magnitudinem ejus determinare, & Ordinatarum situm invenire.

Inveniatur prima diameter Aa, ipsi Bb conjugata per Corollarium præcedens, fiat etiam $AP \times Pa : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ vel $\overline{Cb}^2$; & *erit Bb magnitudo secundæ diametri Bb, & Ordinatæ ejus diametro Aa parallelæ erunt. * Art. 88. & 126.

PROPOSITIO VII.

PROBLEMA.

162. AD *datam Sectionem Conicam, & a puncto* T *extra eam dato, duas tangentes* TM, TM, *ducere.* Fig. 63, 64. & 65.

PRO PARABOLA.

DUCTA per punctum datum T *diametro, quæ Parabolæ in puncto A occurrat, sumptâque parte AP ipsi AT æquali, agatur *per punctum P recta MM Ordinatis parallela, Parabolæ in punctis M & M *occurrens; jungantur que TM, TM, & erunt rectæ TM, TM, *tangentes quæsitæ. * Art. 155. * Art. 158. * Art. 151. * Art. 25. & 26.

PRO-

Pro Ellipsi.

* Art. 151. Ducta per punctum datum T * diametro A*a*, sumtâque CP tertia proportionali ad CT, CA; agatur per punctum P recta MM Ordinatis parallela, Ellipsi in duo-
* Art. 151. bus punctis M & M * occurrens; junganturque TM, TM,
* Art. 65. & 66. & * erunt rectæ TM, TM, tangentes quæsitæ.

Pro Hyperbola et Hyperbolis oppositis.

Ducta (Fig. 65.) per punctum datum T diametro
* Art. 161. A*a* (cujus magnitudo * determinanda est, si fuerit secunda diameter) sumatur CP tertia proportionalis ad CT, CA (ex eâdem parte puncti dati T, quoad centrum, si modo punctum illud in angulo ab Asymptotis facto contineatur; & ex parte oppositâ, quando punctum T in alterutro angulorum, qui deinceps sunt, inveniatur) agaturque per punctum P recta MM Ordi-
Art. 151. natis parallela, Hyperbolæ vel Hyperbolis oppositis * occurrens in punctis M & M; junganturque TM, TM; &
* Art. 131. erunt TM, TM, * tangentes quæsitæ.

Si punctum datum sit centrum, Asymptoti CG, C*g*,
* Art. 118. erunt * tangentes, quas ducere liceret per articulum 157. Quòd si punctum datum S in altero Asymptotôn C*g* fuerit, bisecetur CS in H, & ducatur recta HM alteri
* Art. 112. Asymptoto CG parallela, Hyperbolæ in puncto M * oc-
* Art. 115. currens, & jungatur SM; & * erit SM una tangentium quæsitarum, Asymptotos autem C*g* altera.

COROLLARIUM.

163. QUONIAM recta MPM Ordinatis parallela Sectioni *occurrat in duobus punctis M & M a puncto P aequidistantibus, sequitur duas solummodo tangentes duci posse a puncto T extra sectionem dato; unde si per punctum concursûs T duarum tangentium TM, TM, agatur diameter TP, ea bifariam secabit rectam MPM. Et vice versâ, si diameter TP rectam MPM puncta contactuum jungentem bifariam secet, ea per punctum concursus T transibit. * Art. 151.

PROPOSITIO VIII.

PROBLEMA.

164. DATIS *Conicae Sectionis Diametro, Parametro ejus, & Ordinatarum situ; cognito etiam in Hyperbola, si data diameter fuerit prima vel secunda; describere modo simplici & aequabili tres Conicas Sectiones.*

PRO PARABOLA.

SIT HAL triangulum isosceles, cujus alterum crus AH super diametrum datam AP ex utraque verticis parte productam constituatur, alterum vero AL super tangentem indefinitam LAL per punctum A ductam. Jam si fingamus basim HL motu sibimet parallelo ita ferri, ut una ipsius extremitas L rectam indefinitam LM ipsi AP parallelam secum rapiat, altera vero H, rectam FH ipsi AL parallelam, diametri autem AP parametro aequalem; ita tamen, ut ipsius FH extremitas F secum ferat rectam FA ipsi HL parallelam, & circa punctum Fig. 66.

fixum

fixum A mobilem; dico, continuam rectarum FA, LM, intersectionem M (dum recta HL in angulo HAL, & ejus verticali moveatur) Parabolam quæsitam designare.

Actâ enim ad Diametrum AP Ordinatâ MP, erunt triangula AHF, APM, similia; ideoque erit, AH vel

* 16. El. 1. AL, vel PM : HF :: AP : PM; ergo * $\overline{PM}^2 = AP \times HF$.

* Art. 7 & 23. unde * erit punctum M ad Parabolam.

Notandum est autem, punctum H ultra diametri AP verticem A cadere debere, quando puncta F, L, ex utrâque parte illius diametri cadunt.

Pro aliis Sectionibus.

Fig. 67, 68. Eadem erit constructio ac prior, excepto quòd recta LM circa alteram extremitatem *a* datæ diametri A*a* mobilis esse debet, quum tamen in Parabola ei parallela sit. Ponitur datam diametrum in Hyperbola esse primam; si

* Art. 123. enim esset secunda, facile * inveniretur prima ei Conjugata, ita & Parameter.

Actâ enim MP ad diametrum A*a* Ordinatâ, triangula

* 4. El. 6. *a*PM, *a*AL, & APM, AHF erunt * similia.

Unde *a*P : PM :: *a*A : AL vel AH.

Et AP : PM :: AH : HF.

Ideoque $aP \times AP : \overline{PM}^2 :: aA \times AH : AH \times HF :: aA : HF$,

* Art. 49. 63. & 89. 126. Ergo $aP \times AP : \overline{PM}^2 :: aA : HF$, ideoque punctum M * est ad Sectionem.

Notandum est, puncta H, *a*, ex utraque parte puncti A in Ellipsi cadere debere; & ex eâdem, quando in Hyperbola puncta F, L, ex utrâque parte diametri A*a* cadunt.

PRO-

PROPOSITIO IX.

Theorema.

165. Si *duæ rectæ* MN, AR, *ad Sectionem Conicam terminatæ, & sibi mutuo occurrentes in puncto aliquo* P, *parallelæ fuerint duabus rectis positione datis; dico rectangulum* MP × PN *esse semper ad rectangulum* AP × PR *in ratione datâ, quocunque in puncto Sectionis cadant rectæ* MN, AR. Fig. 69, 70, 71, 72.

Sint enim in Parabola rectæ contingentes CB, EB, sibi mutuo occurrentes in puncto B, rectis etiam MN, AR, parallelæ, dico esse semper, Fig. 69.

$$MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2.$$

Bisectâ enim MN in puncto G, ducatur *diameter CG; & per verticem C acta CB ipsi MN parallela, sectionem in C *continget: ductâ eodem prorsus modo tangente EB ipsi AR parallela, productâ etiam, usque dum diametro CG in puncto K occurrat; demissâque per punctum contactûs E recta EL ad diametrum CG Ordinata, erit *KC = CL; ideoque KB = BE, ob parallelas CB, LE. Porro ducatur Ordinata AD, ita & recta AF ipsi CG parallela; & sit KB vel BE = m; BC = n; CK = e; Diametri CG parameter CH = p; AP = x; PM = y; AD = r; CD = s. * Art. 155. * Art. 11, & 24. * Art. 25, & 26.

His positis, triangula KBC, APF erunt *similia, unde, * Art. 15.

$$KB : BC :: AP : PF; \text{ Hoc est } m : n :: x : \frac{nx}{m} = PF.$$

Eadem ratione erit AF vel DG $= \frac{ex}{m}$.

Unde CG $= \frac{ex}{m} + s$; Et GM vel GN $= y + \frac{nx}{m} + r$.

Et PN $=$ GN $+$ GP $= y + \frac{2nx}{m} + 2r$,

Unde MP $\times$ PN $= yy + \frac{2nx}{m}y + 2ry$,

Et $\overline{GM}^2 = yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr$.

* Art. 8. & 23. Jam verò *CD : CG :: $\overline{AD}^2 : \overline{GM}^2$,

Hoc est $s : \frac{ex}{m} + s :: rr : rr + \frac{err}{ms}x = \overline{GM}^2$;

Unde $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = rr + \frac{err}{ms}x$,

* Art. 22. Sed*$\overline{AD}^2 = rr =$ CD $\times$ CH $= ps$; ergo $rr + \frac{err}{ms}x = rr + \frac{ep}{m}x$,

Ideoque $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = rr + \frac{ep}{m}x$,

Vel $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = 0$.

Quæ æquatio omnibus Parabolæ punctis æqualiter convenit, modò recta AR supra diametrum CG cadat, & punctum concursus P inter puncta A & R inveniatur.

Jam si in hac æquatione ponatur $y = 0$; tum, deletis omnibus terminis, ubi y occurrit, evadet hæc æquatio,

$$\frac{nn}{mm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = 0,$$

Unde dividendo per mx, erit

$$\frac{x}{mm} + \frac{2r}{mn} - \frac{ep}{mnn} = 0. \quad \text{Vel } x = \frac{emp}{nn} - \frac{2mr}{n}.$$

* Hyp. Sed, quoniam *y vel PM $= 0$, erit AP vel $x =$ AR.

Ac

Ac proinde $AR = \frac{emp}{nn} - \frac{2mr}{n}$;

Sed $PR = AR - AP = \frac{emp}{nn} - \frac{2mr}{n} - x$.

Unde $AP \times PR = \frac{emp}{nn}x - \frac{2mr}{n}x - xx$.

Sed $yy + \frac{2nx}{m}y + 2ry : \frac{emp}{nn}x - \frac{2mr}{n}x - xx :: nn : mm$,

nam extremis & mediis in se mutuo ductis elicitur æquatio præcedens, unde

$$MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2.$$

Et cum tangentes CB, BE, eædem semper maneant, quacunque in Sectionis parte cadant rectæ MN, AR, ipsis CB, EB, parallelæ, liquet esse semper

$$MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2.$$

Fieri quidem potest, ut pro vario rectarum MN, AR, situ varii contingant casus; sed cum eorum demonstratio eadem sit, exceptis quibusdam rectis, terminisque evanescentibus, non necesse est, ut in his fusius explicandis diutius morer.

PRO ALIIS SECTIONIBUS.

DUCTIS semidiametris CO, CB, ipsis MN, AR, parallelis, dico esse semper Fig. 70 & 71, 72.

$$MP \times PN : AP \times PR :: \overline{CO}^2 : \overline{CB}^2.$$

Ductâ enim * per medium ipsius MN diametro CG, cujus Ordinata sit MN; demittantur super CG ex punctis A & B, rectæ AD, BE, ipsi MN parallelæ, & agatur AF ipsi CG parallela. * Art. 155.

Sit $CB = m$; $BE = n$; $CE = e$; $CK = t$; $CO = c$; $AP = x$; $PM = y$; $AD = r$; $CD = s$;

* Art. 15. His positis, erit, ob similia * triangula CBE, APF,

Fig. 71, 72. $PF = \frac{nx}{m}$, AF vel $DG = \frac{ex}{m}$. Unde in Hyperbola & Hyperbolis oppositis, erit. $CG = DG \pm DC = \frac{ex}{m} \pm s$;

GM vel $GN = MP + PF - AD = y + \frac{nx}{m} - r$,

Et $PN = GN + GP = y + \frac{2nx}{m} - 2r$,

Ac proinde $MP \times PN = yy + \frac{2nx}{m}y - 2ry$;

Et $\overline{GM}^2 = yy + \frac{2nx}{m}y - 2ry + \frac{nn}{mm}xx - \frac{2nr}{m}x + rr$.

* Art. 90, & 126. Jam vero * $\overline{CD}^2 \mp \overline{CK}^2 : \overline{CG}^2 \mp \overline{CK}^2 :: \overline{AD}^2 : \overline{GM}^2$,

Hoc est, $ss \mp tt : \frac{eexx}{mm} \pm \frac{2esx}{m} + ss \mp tt :: rr : \overline{GM}^2$,

Unde $\overline{GM}^2 = \frac{eerrxx \pm 2emrrsx}{mmss \mp mmtt} + \frac{rrss \mp rrtt}{ss \mp tt}$,

Hoc est, $\overline{GM}^2 = \frac{eerrxx \pm 2emrrsx}{mmss \mp mmtt} + rr$;

* Art. 90, & 126. Sed * $\overline{AD}^2 : \overline{CD}^2 \mp \overline{CK}^2 :: \overline{CO}^2 : \overline{CK}^2$ vel $\frac{rr}{ss \mp tt} = \frac{cc}{tt}$; unde si, in quantitate $\frac{eerrxx \pm 2emrrsx}{mmss \mp mmtt} + rr$, substituatur $\frac{cc}{tt}$ loco ipsius $\frac{rr}{ss \mp tt}$, erit $\overline{GM}^2 = \frac{cceexx \pm 2ccemsx}{mmtt} + rr$.

Sed $\overline{GM}^2 = yy + \frac{2nx}{m}y - 2ry + \frac{nn}{mm}xx - \frac{2nr}{m}x + rr$,

Unde $yy + \frac{2nx}{m}y - 2ry + \frac{nntt - ccee}{mmtt}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0$.

Sed.

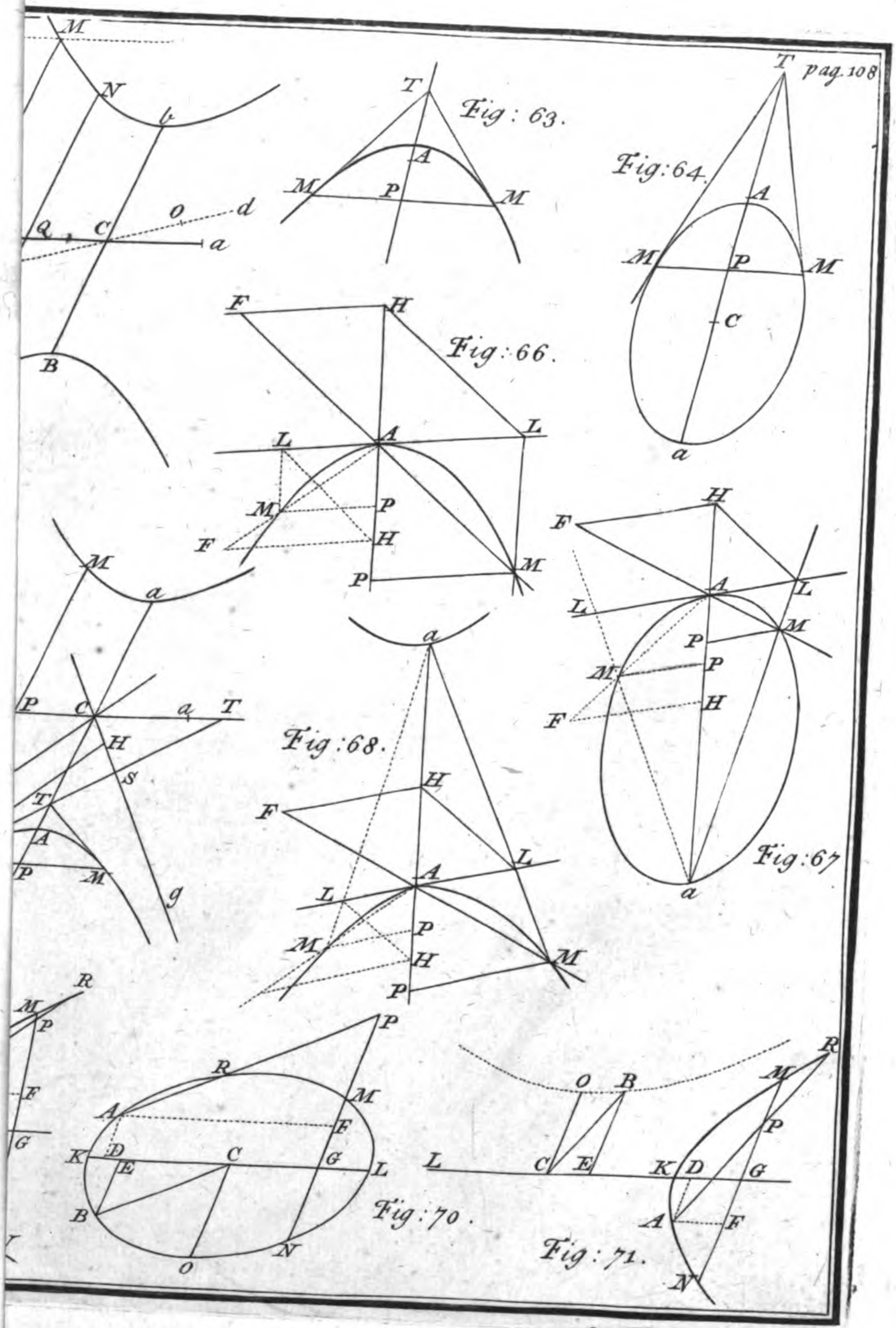

Fig: 63.

Fig: 64.

Fig: 66.

Fig: 67

Fig: 68.

Fig: 70.

Fig: 71.

Sed * $nntt - ccee = cctt$; (nam $\overline{CE}^2 + \overline{CK}^2 : \overline{EB}^2 :: \overline{CK}^2$ * Art. 88, & 126. $: \overline{CO}^2$, vel $ee + tt : nn :: tt : cc$.) Ergo erit $yy + \frac{2nx}{m}y -$

$$2ry + \frac{cc}{mm}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0.$$

Quæ æquatio omnibus Sectionis punctis æquè convenit, modò puncta A & R ex utraque parte diametri CG cadant, punctum autem concursûs P inter puncta A & R inveniatur.

Jam si in hâc æquatione ponatur $y = 0$, evadet (deletis terminis, ubi y occurrit.)

$$\frac{cc}{mm}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0;$$

Unde $x = \frac{2mnrtt \mp 2ccems}{cctt} = AR$, evanescente enim PM vel y erit AP ipsi AR æqualis.

Sed $PR = AR - AP = \frac{2mnrtt \mp 2ccems}{cctt} - x$.

Unde $AP \times PR = \frac{2mnrtt \mp 2ccems}{cctt}x - xx$,

Sed * $\frac{2mnrtt \mp 2ccems}{cctt}x - xx : yy + \frac{2nx}{m}y - 2ry :: mm : cc$, * 16. El. 6.

Vel $AP \times PR : MP \times PN :: \overline{CB}^2 : \overline{CO}^2$,

nam extremis & mediis in se mutuo ductis, eadem elitur æquatio, ac prior; videlicet $yy + \frac{2nx}{m}y - 2ry + \frac{cc}{mm}xx + \frac{2nrtt \mp 2cces}{mtt}x = 0$.

Et cum semidiametri CO, CB, eædem maneant, quâcunque

cunque in Sectionis parte cadant rectæ MN, AR, ipsis CO, CB parallelæ, liquet esse semper

$$MP \times PN : AP \times PR :: \overline{CO}^2 : \overline{CB}^2.$$

Demonstratio pro Ellipsi eadem erit, exceptis quibusdam rectis.

Corollarium I.

Fig. 73. 166. Si duæ rectæ MN, AR ad Sectionem Conicam terminatæ, sibi mutuo occurrant in puncto P; ducantur autem rectæ FG, BD, ipsis MN, AR, parallelæ & ad Sectionem terminatæ, sibi etiam occurrentes in puncto Q; constat esse semper rectangulum $MP \times PN : AP \times PR :: FQ \times QG : BQ \times QD$; ductis enim semidiametris CY, CZ, ipsis MN, AR, parallelis, liquet rectas FG, BD iisdem CY, CZ, esse parallelas; unde erit, $MP \times PN : AP \times PR :: \overline{CY}^2 : \overline{CZ}^2 :: FQ \times QG : BQ \times QD$.

Corollarium II.

167. Si duæ rectæ AR, BD, sibi invicem parallelæ & ad Sectionem Conicam terminatæ, alteri rectæ FG ad eandem Sectionem etiam terminatæ in punctis E & Q occurrant, erit $FE \times EG : AE \times ER :: FQ \times QG : BQ \times QD$.

Si enim fingamus rectam MN in corollario præcedenti cadere super rectam FG, liquet rectangulum $MP \times PN$ fieri $FE \times EG$; ita & $AP \times PR$ fieri $AE \times ER$.

Corollarium III. Pro Circulo.

Fig. 74. 168. Ex Theoremate deducitur nota circuli proprietas; nempe si intra vel extra circulum per punctum quodvis P ducantur rectæ quotcunque AR, MN, HL *&c.* ad circumductum circuli terminatæ, omnia rectangula $MP \times PN$, $AP \times PR$, & $HP \times PL$ *&c.* sibi mutuo

tuo æqualia erunt; ductis enim ſemidiametris CB, CO, CD &c. ipſis AR, MN, HL &c. parallelis, conſtat omnia rectangula eſſe ad invicem, ut quadrata ſemidiametrorum CB, CO, CD &c. ex natura circuli æqualium.

Corollarium IV. Pro Parabola.

169. Si per Parabolæ punctum quodvis A ducatur diameter AF, rectæ MN ad Sectionem utcunque terminatæ occurrens in F; dico rectangulum MF × FN æquari rectangulo ſub AF & ſub parametro CH diametri CG, per medium G ipſius MN ductæ. Fig. 75.

Si enim ponamus rectam AP, in Theoremate, ſuper rectam AF cadere, conſtat PF vel $\frac{nx}{m}$ prorſus evaneſcere, ideoque erit $\frac{n}{m} = o$. Deletis igitur omnibus terminis, ubi $\frac{n}{m}$ inveniatur, in æquatione $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{bmm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = o$, ad Parabolam ſpectante, evadet hæc æquatio, $yy + 2ry - \frac{ep}{m}x = o$; vel $yy + 2ry = \frac{epx}{m}$; ſed eſt * AF $= \frac{ex}{m}$; & CH $= p$; unde AF × CH $= \frac{epx}{m}$. * Art. 165.

Sed eſt etiam MF $= y$; nam evaneſcente PF, eſt MF $=$ MP, & FN $= y + 2r$; unde erit MF × FN $= yy + 2ry$; ideoque MF × FN $=$ AF × CH;

Sed

Sed & hoc facilius demonstrari potest;

* Art. 6. Nam * $\overline{GM}^2 = GC \times CH$; & $\overline{AD}^2$ vel $\overline{GF}^2 = DC \times CH$;

Unde $\overline{GM}^2 - \overline{GF}^2 = GC - DC \times CH = AF \times CH$,

* 5. El. 2. Sed * $\overline{GM}^2 - \overline{GF}^2 = MF \times FN$,

Ergo $MF \times FN = AF \times CH$.

Corollarium V. Pro Parabola.

170. Hinc si in Parabola per duo puncta quæcunque A & B agantur duæ diametri AF, BP, duabus rectis MN, EL, sibi mutuo parallelis, & ad Sectionem terminatis occurrentes in punctis F & P; constat esse $MF \times FN : EP \times PL :: AF : BP$.

* Art. 152. Diameter enim CG per medium ipsius MN ducta transit * etiam per medium ipsius EL; ac proinde $EP \times PL = BP \times CH$.

Et $MF \times FN = AF \times CH$;

Sed $AF \times CH : BP \times CH :: AF : BP$,

Ergo $MF \times FN : EP \times PL :: AF : BP$.

2do. Si recta MN ad Parabolam terminata duabus diametris AF, BK in punctis F & K occurrat, erit,

$MF \times FN : MK \times KN :: AF : BK$;

Nam $MF \times FN : EP \times PL :: AF : BP$,

Et $EP \times PL : MK \times KN :: BP : BK$,

Ex æquo $MF \times FN : MK \times KN :: AF : BK$.

Unde constat *3tio*, si duæ rectæ MN, EL ad Parabolam terminatæ & sibi mutuo parallelæ diametro cuivis BP in punctis K & P occurrant, esse,

$MK \times KN : EP \times PL :: BK : BP$.

Cor-

COROLLARIUM VI. PRO PARABOLA.

171. HINC per data tria puncta A, M, N, describi potest Parabola, cujus diametri AF, CG rectæ positione datæ parallelæ fuerint.

Juncta MN, agatur per tertium punctum A, recta AF rectæ positione datæ parallela, ipsi etiam MN in puncto F occurrens; & per medium ipsius MN agatur GC ipsi AF parallela; fiat deinde $MF \times FN$; $MG \times GN$ vel $\overline{GM}^2$ :: AF : GC; & sit CH tertia proportionalis ad rectas CG, GM; tum *parametro CH, & diametro CG, cujus vertex C, Ordinatæ autem ipsi MN parallelæ fuerint, describatur Parabola; & ea erit quæsita. * Art. 35. & 36.

Transibit *enim per puncta M & N quoniam $CH \times CG = \overline{GM}^2$ vel $\overline{GN}^2$; & per *punctum A, nam $MG \times GN : MF \times FN$ :: CG : FA. * Art. 7 & 23. * Art. 170.

Porro Diametri AF, CG, rectæ positione datæ parallæ sunt; & cum Parabola hac ratione descripta habeat pro diametro rectam CG, cujus vertex sit punctum C, & pro parametro determinatam rectam CH, constat hanc esse solam, quæ per data puncta A, M, N, describi possit.

COROLLARIUM VII. PRO PARABOLA.

172. SI duæ rectæ AR, MN, ad Parabolam terminatæ sibi mutuo occurrant in puncto P; fiat autem $AP \times PR : MP \times PN$ :: $\overline{AP}^2 : \overline{PF}^2$, & jungatur AF; dico hanc rectam esse diametrum. Fig. 69.

Ductis enim contingentibus CB, EB, ad rectas MN,

AR parallelis, & per punctum C diametro CG ipsi EB
* Art. 165. occurrente in K; erit * $\overline{EB}^2$ vel $\overline{KB}^2 : \overline{BC}^2 :: AP \times PR : MP \times PN$;

* Hyp. Sed * $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$,

Ergo $KB : BC :: AP : PF$. Unde triangula KBC, APF
* 6. El. 6. erunt * similia, ideoque latera AF, KC, sibi mutuo parallela; sed CG est diameter, ergo AF ipsi CG parallela
* Def. 7. 1. * erit etiam diameter.

Corollarium VIII. Pro Parabola.

173. Ex præcedenti consectario deducitur methodus describendi Parabolam per quatuor puncta data, A, M, R, N.

Nam si quatuor illa puncta duabus rectis AR, MN, sibi mutuo in P occurrentibus jungantur, & si fiat $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$; ducatur autem recta AF,
* Art. 171. describaturque * per puncta A, M, N, Parabola, cujus diametri rectæ AF parallelæ sint, liquet hanc esse quæsi-
* Art. 165. tam; necesse est enim recta AP Parabolæ ita * occurrat in R ut sit $AP \times PR : MP \times PN :: \overline{EB}^2$ vel $\overline{KB}^2 : \overline{BC}^2$.

* Art. 172. Sed * $\overline{KB}^2 : \overline{BC}^2 :: \overline{AP}^2 : \overline{PF}^2$,

Unde $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$.

Fig. 76. Quòd si punctum F ex alterâ parte puncti P caperetur, describi posset etiam alia Parabola, quæ per quatuor puncta data transiret.

Sin vero punctum F supra alterutrum punctorum M, N, caderet, una tantum describi posset Parabola, quæsitis conditionibus.

Quando

Quando autem bina puncta F, F, super bina puncta M, N, cadant, nulla omnino describi potest Parabola; ex eo quod in hoc casu Parabolæ diameter AF per duo ejus puncta transiret, quod * fieri nequit. * Art. 11.

COROLLARIUM IX.

PRO HYPERBOLA VEL HYPERBOLIS OPPOSITIS.

174. SI recta MN ad Hyperbolam vel Hyperbolas Fig. 77, 78. oppositas terminata, rectæ etiam positione datæ parallela, Asymptoto CB in puncto Q occurrat; ducatur autem per punctum quodvis A recta AP eidem Asymptoto parallela, ipsi MN in puncto P occurrens; dico rectangulum MP × PN esse semper ad rectangulum 2AP × PQ in ratione datâ, quacunque in Sectionis parte cadant rectæ MN, AP.

Si enim in Theoremate (Fig. 71, 72) ponamus semidiametrum CB fieri Asymptoton, constat latera trianguli CBE tum infinita evadere; ideoque si in figuris 77, 78, per terminum K diametri LK per medium ipsius MN ductæ, agatur KS ipsi MN parallela, Asymptoto CB occurrens in S, constitutum erit triangulum CKS, cujus latera sint finita, quodque triangulo CBE * simile erit. * Art. 15.

Unde CK : KS vel * CO :: CE : EB, * Art. 121.
Hoc est $t : c :: e : n$, unde * $ce = nt$ ac proinde si, loco * 16. El. 6. ipsius ce, substituatur valor ejus nt, in æquatione

$$yy + \frac{2nx}{m}y - 2ry + \frac{nntt - ccee}{mmtt}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0,$$

ad Hyperbolam spectante, evadet

$$yy + \frac{2nx}{m}y - 2ry - \frac{2nrt \mp 2cns}{mt}x = 0,$$

$$\text{Vel } yy + \frac{2nx}{m}y - 2ry = \frac{2nrt \pm 2cns}{mt}x;$$

Productâ igitur rectâ AD, si opus, usque dum Asymptoto CB occurrat in H; erunt triangula CKS, CDH
* 4. El. 6. * similia, unde

$$CK : KS :: CD : DH, \text{ vel } t : c :: s : \frac{cs}{t} = DH.$$

* 34. El. 1. Ideoque AH vel * $PQ = AD \pm DH = r \pm \frac{cs}{t} = \frac{rt \pm cs}{t}$.

$$\text{Unde } 2AP \times PQ = \frac{2rt \pm 2cs}{t}x.$$

* 16. El. 6. Sed * $yy + \frac{2nx}{m}y - 2ry : \frac{2rt \pm 2cs}{t}x :: n : m$;

Vel $MP \times PN : 2AP \times PQ :: EB : CB$, nam extremis & mediis in se invicem ductis, evadet æquatio

$$yy + \frac{2nx}{m}y - 2ry = \frac{2nrt \pm 2cns}{mt}x;$$

Et cum rectæ KS, CS, eædem maneant, quacunque in Sectionis parte cadant rectæ MN, AP (ex eo quod diameter LK per medium ipsius MN ducta, transit
* Art. 152. * etiam per medium rectarum omnium ipsi MN parallelarum, & ad Sectionem terminatarum) erit semper $MP \times PN : 2AP \times PQ :: EB : CB :: KS : CS$.

* Fig. 77. Sed & hoc alio modo demonstrari potest.

Sit $CK = t$; KS vel $CO = c$; $CS = m$; $CD = s$; AD vel $DI = r$; $AP = x$; $PM = y$;

* Art. 15. Et erunt triangula CSK, APF * similia,

Unde

Unde CS : SK :: AP : PF,

Vel $m : c :: x : \frac{cx}{m} = PF$; eodem prorsus modo ostenditur AF vel $DG = \frac{tx}{m}$,

Ideoque GM vel $GN = PM + PF - GF = y + \frac{cx}{m} + r$;

Et $CG = DG + CD = \frac{tx}{m} + s$.

Porro ob triangula CKS, CDH, CGQ * similia, erit, CK : KS :: CD : DH; *2. El. 6.

Scilicet $t : c :: s : \frac{cs}{t} = DH$;

Et CD : DH :: CG : GQ,

$$s : \frac{cs}{t} :: \frac{tx}{m} + s : \frac{cx}{m} + \frac{cs}{t} = GQ;$$

Sed * $MQ \times QN = \overline{GQ}^2 - \overline{GM}^2$; *6. El. 2.

Unde $MQ \times QN = \frac{2ccsx}{mt} + \frac{ccss}{tt} - yy - \frac{2cx}{m}y + 2ry + \frac{2crx}{m} - rr$.

Est etiam * $MQ \times QN = AH \times HI = \overline{DH}^2 - \overline{DI}^2$, * Art. 105.

Ergo $MQ \times QN = \frac{ccss}{tt} - rr$;

Unde $\frac{2ccsx}{mt} + \frac{ccss}{tt} - yy - \frac{2cx}{m}y + 2ry + \frac{2crx}{m} - rr = \frac{ccss}{tt} - rr$,

Vel $yy + \frac{2cx}{m}y - 2ry = \frac{2ccsx}{mt} + \frac{2crx}{m}$,

Ergo * $yy + \frac{2cxy}{m} - 2ry : \frac{2csx}{t} + 2rx :: c : m$, * Art. 43.

Scilicet $MP \times PN : 2AP \times PQ :: KS : CS$;

Eadem

Eadem prorſus eſt demonſtratio pro Hyperbolis oppoſitis, ſignis ſolummodo quibuſvis mutatis.

Corollarium X.

Pro Hyperbola et Hyperbolis oppositis.

175. Ex Corollario præcedenti ſequitur, *1mo,* ſi duæ rectæ MN, HG, ad Hyperbolam vel Hyperbolas oppoſitas terminatæ, Aſymptoto CS in punctis I & Q occurrant; ducantur autem per duo quævis Sectionis puncta A & B rectæ AP, BD, Aſymptoto CS parallelæ, ipſis MN, HG, in punctis P & D occurrentes; liquet eſſe ſemper $MP \times PN : 2AP \times PQ :: HD \times DG : 2BD \times DI$, Ideoque $MP \times PN : AP \times PQ :: HD \times DG : BD \times DI$.

2do. Si duæ rectæ MN, HG, ſibi mutuo parallelæ, & ad Hyperbolam vel Hyperbolas oppoſitas terminatæ Aſymptoto CS in punctis Q & I occurrant; ducatur autem ex Sectionis puncto quovis A recta AO, ipſi CS parallela, ipſis MN, HG, in punctis P & O occurrens; conſtat, ſi ponamus in præcedenti numero rectam BD ſuper AP cadere, eſſe

$MP \times PN : HO \times OG :: AP \times PQ : AO \times OI :: AP : AO$, Ergo $MP \times PN : HO \times OG :: AP : AO$.

3tio, Si recta HG ad Hyperbolam vel Hyperbolas oppoſitas terminatæ Aſymptoto CS occurrat in I; agantur autem per duo quævis Sectionis puncta A, B, rectæ AO, BD, Aſymptoto CS parallelæ, ipſi HG in punctis O & D occurrentes; erit

$$HO \times OG : HD \times DG :: AO \times OI : BD \times DI.$$

Hoc etiam ſequitur ex numero *1mo,* ſi rectam MN ſuper HG cadere fingamus.

Cor-

COROLLARIUM XI.

176. Si fingamus rectam BD, quæ Sectioni Conicæ in duobus punctis B & D occurrit, motu sibimet parallelo eo usque ferri, dum fiat tangens LS; liquet puncta occursûs B & D, tum in unum coitura ad punctum contactûs L; ideoque punctum contactûs pro duobus intersectionis punctis in se mutuo cadentibus haberi potest. Fig. 75.

Hoc posito; constat 1mo, si duæ contingentes KS, LS, sibi mutuo occurrant in puncto S; agantur autem duæ aliæ rectæ MN, AR, contingentibus parallelæ, ad Sectionem terminatæ, & sibi mutuo occurrentes in puncto P; esse $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$; hoc enim in Theoremate de Parabola jam ostensum est. Pro aliis autem Sectionibus, si * in primo consectario fingamus, rectam FG super tangentem KS cadere, & BD super LS, liquet puncta intersectionis F & G, in puncto contactus K conventura, ita & puncta B, D, in puncto contactûs L; unde rectangulum $FQ \times QG$ evadet quadratum $\overline{KS}^2$, & rectangulum $BQ \times QD$ erit quadratum $\overline{LS}^2$; ac proinde erit $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$. * Art. 166.

2do. Si in Ellipsi vel in Hyperbolis oppositis ducatur contingens TX ipsi KS parallela rectæ SL in puncto X occurrens, eadem ratione ac in numero præcedenti ostenditur, esse $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$; cadente enim recta FG super tangentem TX, puncta intersectionis F & Q convenient in T; ita & puncta B & D convenient in L; ideoque $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$.

Unde

Unde ſequitur, ſi tangentes duæ KS, TX, ſibi invicem parallelæ tangenti tertiæ LS occurrant in punctis S, X, erit KS : LS :: TX : LX,

$$\text{Nam } MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2,$$

$$\text{Et } MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2. \text{ Ergo \&c.}$$

3tio. Si in Ellipſi vel Hyperbolis oppoſitis duæ contingentes KS, LS ſibi mutuo occurrant in S; ducantur autem duæ ſemidiametri CY, CZ, contingentibus parallelæ; dico eſſe ſemper KS : LS :: CY : CZ;

$$\text{Nam } MP \times PN : AP \times PR :: \overline{CY}^2 : \overline{CZ}^2,$$

$$\text{Et } MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2,$$

$$\text{Ergo } \overline{CY}^2 : \overline{CZ}^2 :: \overline{KS}^2 : \overline{LS}^2, \text{ Et } CY : CZ :: KS : LS.$$

4to. Si duæ rectæ AR, FG, ad Sectionem Conicam terminatæ, duabus contingentibus KI, LO, occurrant in punctis I & O, dico eſſe ſemper FO × OG : $\overline{LO}^2$ :: $\overline{KI}^2$: AI × IR, ſi enim fingamus rectam BD in corollario * primo fieri tangentem LO, & rectam MN fieri tangentem KI, res patet.

* Art. 166.

5to. Si duæ parallelæ AR, BD, ad Sectionem Conicam terminatæ contingenti KH in punctis I & H occurrant, erit

$$\overline{KI}^2 : AI \times IR :: \overline{KH}^2 : BH \times HD,$$

Vel $\overline{KI}^2 : \overline{KH}^2$ :: AI × IR : BH × HD. Hoc ex ſecundo conſectario deduci poteſt, ſi fingamus rectam FG ſuper tangentem KH cadere.

6to. Si in numero præcedenti ponamus Sectionem Conicam fieri Hyperbolam, cujus Aſymptotos ſit tangens

tangens HK, liquet rectangula $BH \times HD$ & $AI \times IR$ fore æqualia, punctum enim contactûs K infinitè *distat a punctis H & I; ideoque rectæ infinitæ HK, IK, finitâ quantitate HI inter se differentes, pro æqualibus rectis sunt habendæ. * Art. 118.

7mo. Si duæ contingentes KS, LS, sibi mutuo occurrant in puncto S, recta autem AR ad Sectionem terminata, alteri etiam contingentium LS parallela, alteri KS in puncto I occurrat; dico esse $\overline{KI}^2 : AI \times IR :: \overline{KS}^2 : \overline{LS}^2$; hoc constabit ex corollario *secundo, si ponamus rectas FG, BD, super tangentes KS, LS, cadere. * Art. 167.

8vo. Si in Ellipsi vel Hyperbolis oppositis duæ contingentes KI, TV, sibi invicem parallelæ, occurrant in punctis I & V rectæ AR ad Sectionem in A & R terminatæ; dico esse $\overline{KI}^2 : AI \times IR :: \overline{TV}^2 : RV \times VA$; hoc etiam ex corollario secundo deducitur, si parallelæ MN, FG, super tangentes TV, KI cadere ponantur.

9no. Si rectæ HX, PI, sibi invicem parallelæ & Sectionem secantes in punctis X, Y, P, C, tangenti HI occurrant in punctis H & I; erit $\overline{HA}^2$ ad $\overline{AI}^2$ in ratione composita ex ratione rectanguli $XH \times HY$ ad rectangulum $BH \times HD$, & ex ratione rectanguli $BH \times HD$ ad rectangulum $PI \times IC$. Fig. 80.

Nam *$\overline{HA}^2 : \overline{AI}^2 :: XH \times HY : PI \times IC$. * Num. 5.
Sed *$XH \times HY$ est ad $PI \times IC$ in ratione compositâ ex ratione ipsius $XH \times HY$ ad $BH \times HD$ & ex ratione $BH \times HD$ ad $PI \times IC$. Ergo &c. * Art. 130.

10mo. Hinc si ducantur parallelæ XH, PI, tangenti HI in punctis H & I occurrentes; agatur autem ex pun-

cto H recta HB Sectionem in punctis D & B secans, & ipsi PI productæ occurrens in puncto G; erit $XH \times HY : BH \times HD :: CG \times GP : DG \times GB$.

* Num. 3. & 7. Ductis enim semidiametris EA, EL, EZ, ad ipsas PI, HB, HA, parallelis, erit *

$$XH \times HY : \overline{HA}^2 :: \overline{EA}^2 : \overline{EZ}^2,$$

$$\text{Sed } \overline{HA}^2 : BH \times HD :: \overline{EZ}^2 : \overline{EL}^2,$$

$$\text{Ex æquo } XH \times HY : BH \times HD :: \overline{EA}^2 : \overline{EL}^2;$$

* Art. 165. Est autem * $CG \times GP : DG \times GB :: \overline{EA}^2 : \overline{EL}^2$,

$$\text{Ergo } XH \times HY : BH \times HD :: CG \times GP : DG \times GB.$$

Fig. 81. 11*mo*. Si rectæ HA, PL, Sectionem contingentes, concurrant in G; agatur autem recta IY tangenti alteri PL parallela, Sectioni in punctis X & Y, tangenti autem alteri in puncto I occurrens, & ducatur utcunque recta IL Sectionem in duobus punctis C & D secans, ipsi AP per puncta contactuum A & P ductæ occurrens in S; dico esse $XI \times IY : \overline{LP}^2 :: CI \times ID : CL \times LD$.

* Num. 3. & 7. Ductis enim semidiametris EK, EN, EM ad ipsas IY, IL, HA, parallelis, * erit

$$XI \times IY : \overline{IA}^2 :: \overline{EK}^2 : \overline{EM}^2,$$

$$\text{Et } \overline{IA}^2 : CI \times ID :: \overline{EM}^2 : \overline{EN}^2,$$

$$\text{Ex æquo } XI \times IY : CI \times ID : \overline{EK}^2 : \overline{EN}^2;$$

$$\text{Sed } \overline{LP}^2 : CL \times LD :: \overline{EK}^2 : \overline{EN}^2,$$

$$\text{Ergo } XI \times IY : CI \times ID :: \overline{LP}^2 : CL \times LD.$$

Fig. 75. 12*mo*. Sint in Parabola duæ rectæ MN, CH, sibi invicem

vicem parallelæ, quarum altera CH Sectionem in puncto C contingat, altera vero MN ad Parabolam terminetur; ducatur autem per duo quævis Sectionis puncta duæ diametri AF, BO, ipsis MN, CH, in punctis F & O, occurrentes; liquet, si in consectario * quinto, numeris 1*mo*, 2*do*, ponamus rectam EL super tangentem CH cadere, esse 1*mo*. $MF \times FN : \overline{CO}^2 :: AF : BO$. 2*do*. Productâ FA, usque dum tangenti CH in puncto Q occurrat; esse etiam $MF \times FN : \overline{CQ}^2 :: AF : AQ$. * Art. 170.

13*mo*. Sint rectæ MN, KT, sibi invicem parallelæ; quarum una KT Hyperbolam in K contingens, Asymptoto CI occurrat in puncto S; altera vero MN, ad Hyperbolarum oppositarum alterutram terminata eidem Asymptoto in puncto Q occurrat; agantur autem per duo quævis Sectionis puncta A, B, rectæ AP, BT, Asymptoto CI parallelæ, ipsis MN, KT, in punctis P & T occurrentes; liquet, si ponamus in tribus numeris Corollarii decimi, secantem GH super tangentem KT cadere, esse, 1*mo*. $MP \times PN : \overline{KI}^2 :: AP \times PQ : BT \times TS$. 2*do*. Producta PA, usque dum ipsi KT in puncto R occurrat, esse $MP \times PN : \overline{KR}^2 :: AP : AR$. 3*tio*. $\overline{KT}^2 : \overline{KR}^2 :: BT \times TS : AR \times RS$. Fig. 79.

14*mo*. Si in Hyperbolis oppositis duæ contingentes KR, LF, sibi mutuo parallelæ, Asymptoto CS in punctis S & V occurrant; agantur autem per duo quævis Sectionis puncta A, B, rectæ AR, BF, Asymptoto CS parallelæ, contingentibus etiam in punctis R & F occurrentes; dico esse, 1*mo*. $\overline{KR}^2 : \overline{LF}^2 :: AR \times RS : BT \times FV$. 2*do*. $\overline{KR}^2 : \overline{LE}^2 :: AR : AE$, quod constabit, si pona-

 namus

mus secantes MN, GH, super rectas contingentes KR, LF, cadere.

PROPOSITIO X.

Theorema.

Fig. 82. & 83. 177. Si *in Ellipsi vel in duabus Hyperbolis oppositis, quarum centrum* C, *focorum distantia* FG, *& axis primus* BD, *sumatur* CA *tertia proportionalis ad rectas* CF, CB; *ducatur autem per punctum* A *recta* AP *ipsi* BD *normalis, agaturque ex Sectionis puncto quovis* M *recta* MF *ad focum* F, *&* MP *ipsi* AP *perpendicularis*; *dico esse semper* MP : MF :: BD : FG.

Pro Ellipsi.

Fig. 82. Sit CB $= a$; CF $= b$; AF $= c$; PM $= y$; AP $= x$; Parameter ipsius KH $=$ KL.

* 47. El. 1. Unde FQ $=$ AF $-$ PM $= c - y$; & * $\overline{MF}^2 = cc - 2cy + yy + xx$.

Sed DF $= a + b$, & FB $= a - b$.

* Art. 42. Unde DF $\times$ FB * $= \overline{CH}^2 = aa - bb$;

* Art. 48. Sed * $\overline{CH}^2 : \overline{CB}^2 ::$ KH : KL,

Ergo KH : KL :: $aa - bb : aa$.

* Hyp. Porro quoniam * CF : CB :: CB : CA, erit etiam CF : CA :: $\overline{CF}^2 : \overline{CB}^2$, vel $\overline{CB}^2 : \overline{CF}^2 ::$ CA : CF,

Unde $\overline{CB}^2 - \overline{CF}^2 : \overline{CB}^2 ::$ CA $-$ CF : CA,

Sive $aa - bb : aa :: c : \frac{aac}{aa-bb} =$ CA,

Ideoque

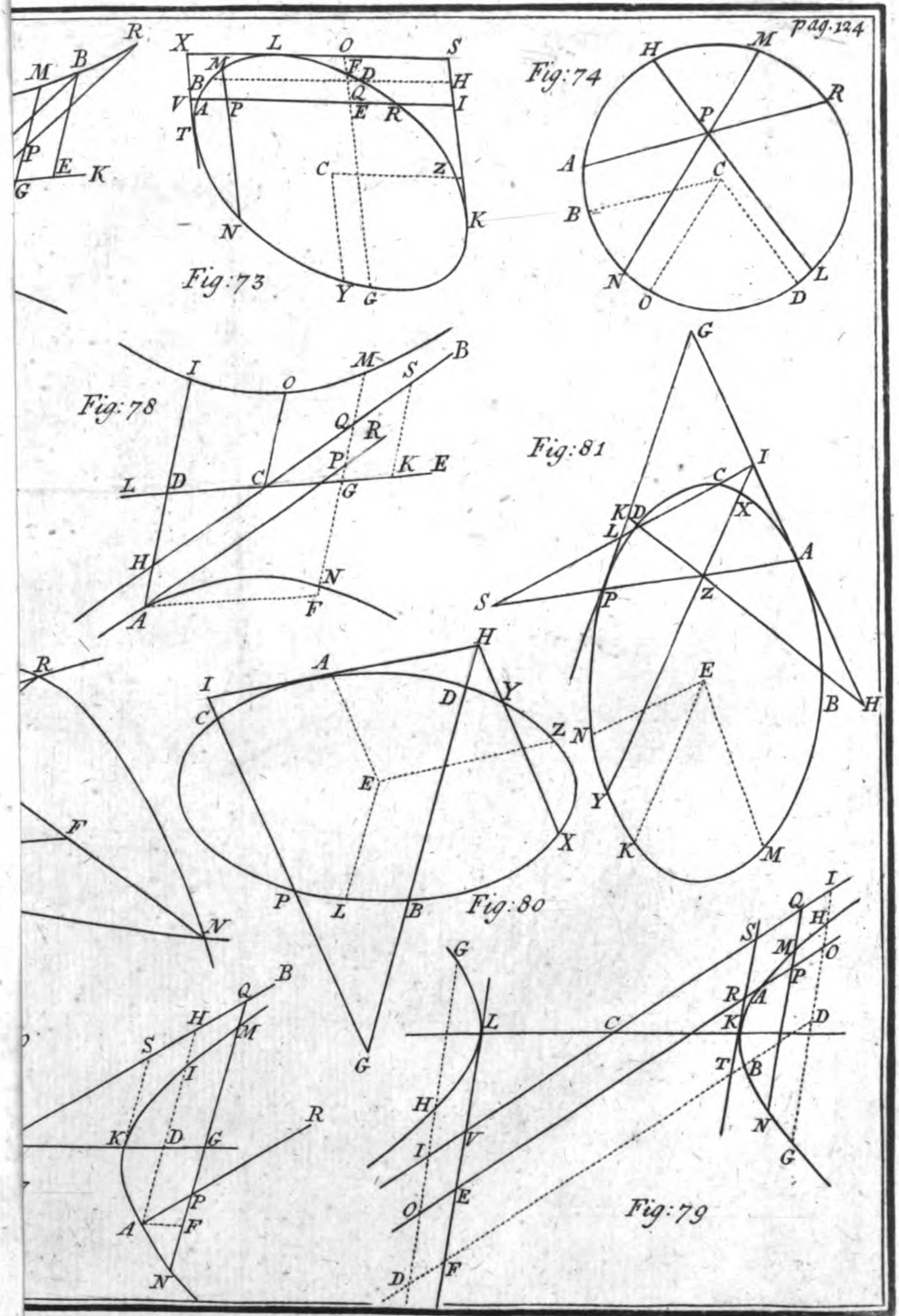

Fig: 73

Fig: 74

Fig: 78

Fig: 81

Fig: 80

Fig: 79

Ideoque erit $EM = AC - AQ = \frac{aac}{aa-bb} - y$.

Rursus quoniam $CF : CB :: CB : CA$, erit etiam * $CB - CF : CF :: CA - CB : CB$,

Id est, $BF : FC :: AB : BC$,

Unde $BC : CF :: AB : BF$,

Ideoque * $DF : FC :: AF : FB$, * 18. El. 5.

Ergo * $DF \times FB = \overline{CH}^2 = CF \times FA$. * 16 El. 6.

Et erit $CF : CH :: CH : FA$,

Ac proinde $\overline{CH}^2 : \overline{CF}^2 :: \overline{FA}^2 : \overline{CH}^2$,

Hoc est, $aa - bb : bb :: cc : \frac{bbcc}{aa-bb} = \overline{CH}^2$;

Sed * $KE \times EH$ vel $\overline{CH}^2 - \overline{CE}^2 : \overline{EM}^2 :: KH : KL$, * Art. 49.

Hoc est, $\frac{bbcc}{aa-bb} - xx : \frac{a^4cc}{\overline{aa-bb}^2} - \frac{2aacy}{aa-bb} + yy :: aa - bb : aa$;

Unde * $\frac{aabbcc}{aa-bb} - aaxx = \frac{a^4cc}{aa-bb} - 2aacy + aayy - bbyy$. * 16. El. 6.

Vel $\frac{a^4cc}{aa-bb} - \frac{aabbcc}{aa-bb} + aaxx + aayy - bbyy - 2aacy = 0$,

Et dividendo æquationem per $aa - bb$, evadet

$\frac{aacc}{aa-bb} - \frac{2aacy}{aa-bb} + \frac{aaxx}{aa-bb} + yy = 0$,

Vel $aacc - 2aacy + aayy + aaxx = bbyy$,

Unde * $yy : cc - 2cy + yy + xx :: aa : bb$, * Art. 43.

Viz. $\overline{MP}^2 : \overline{MF}^2 :: \overline{CB}^2 : \overline{CF}^2$,

Sed $\overline{CB}^2 : \overline{CF}^2 :: \overline{DB}^2 : \overline{FG}^2$;

Ergo $MP : MF :: DB : FG$.

Demonstratio eadem prorsus erit pro duabus Hyperbolis oppositis, signis quibusdam mutatis.

COR-

Corollarium I.

178. Hinc $AB : BF :: DB : GF$,

* Hyp. Nam * $CA : CB :: CB : CF$, unde $AB : CB :: BF : CF$. ideoque $AB : BF :: 2CB : 2CF :: DB : GF$.

Corollarium II.

Fig. 82. 179. Hinc etiam, si recta AB major sit quam BF, Sectio erit Ellipsis, ex eo quod axis primus in hoc casu major sit quam distantia focorum G & F; sin autem AB minor sit quam BF, ac proinde axis primus BD minor quam distantia focorum F, G, Sectio erit Hyperbola; quod si AB æqualis fuerit ipsi BF, Sectio tum * erit Parabola.

Fig. 83.
Fig. 84.
* Art. 1.

Scholium.

180. Recta indefinita AP vocatur Directrix non ad Parabolam tantum, sed & ad Ellipsin vel Hyperbolas oppositas.

Corollarium III.

Fig. 85, 86. 181. Si in Sectione Conica duo quævis puncta M & N rectâ MN Directrici in puncto C occurrente jungantur; a foco autem F agantur rectæ FM, FN, FC; dico rectam FC bisecare angulum NFH (complementum scilicet anguli NFM ad duos rectos) quando puncta M, N, in Parabola, Ellipsi vel in Hyperbola cadunt; angulum vero NFM, quando puncta illa in Hyperbolis oppositis cadunt.

Ductis enim MP, NQ, Directrici CP normalibus, recta autem ND ipsi MF parallela; triangula MPC, NQC & MFC, NDC erunt similia, unde $MP : NQ :: MC : NC :: MF : ND$, & $MP : MF :: NQ : ND$;

Sed $MP : MF :: NQ : NF$; ideoque $ND = NF$; unde in primo casu angulus NDF, vel ob parallelas CFH, æquatur angulo CFN; & in secundo FDN, vel ei alternus CFM, æquatur angulo CFN.

Fig. 85.
Fig. 86.

LIBER

LIBER QUINTUS DE SECTIONIBUS CONICIS INTER SE MUTUO COMPARATIS.

Lemma I.

182. Quantitates, *ut & quantitatum rationes, quæ ad æqualitatem tempore quovis finito constanter tendunt, & ante finem temporis illius propius ad invicem accedunt, quam pro data quavis differentia, fiunt ultimo æquales.*

Si negas; fiant ultimo inæquales, & sit earum ultima differentia D; ergo nequeunt propius ad æqualitatem accedere, quam pro data differentia D; contra Hypothesin.

Lemma II.

183, Sit *Arcus* MN *curvæ cujusvis* ABG *portio infinitè parva, hoc est, minor quavis datâ: jam si per terminos ipsius* MN *ad axem vel diametrum* AC *Ordinatim applicentur* MP, NQ; *agantur autem* MR, NS, *ipsi* AC *parallelæ; dico parallelogramma* PQRM *vel* PQNS *haberi posse pro spatio* PQNM *sub Ordinatis* PM, NQ, *rectâ* PQ, *& arcu* MN *comprehenso.* Fig. 87.

Omnia Curvæ cujusvis puncta ab ejus Diametro vel continuò recedunt, vel ad eam continuo accedunt; vel denique Curva omnis ex pluribus constat portiunculis, quarum

quarum aliæ a Diametro magis magisque recedunt, aliæ ad eam magis magisque accedunt; fieri enim non potest, ut Curvæ pars ulla a Diametro æquidistet.

His præmissis, sit *1mo* arcus MN portio Curvæ AMB ab ejus Diametro magis magisque recedentis; sumatur ex parte puncti N arcus MO finitæ magnitudinis, ductâque Ordinata OF ipsi MP parallela, agantur rectæ OD, ME Diametro AC parallelæ; & erit spatium curvilineum PFOM majus parallelogrammo inscripto PFEM, circumscripto autem PFOD minus. Jam si punctum O super Curvam versus M moveri fingarur, liquet parallelogrammum MEOD (differentiam scilicet parallelogramorum in arcu OM inscriptorum & circumscriptorum) continuo diminui, adeo ut puncto O in M pervento prorsus evanescat MEOD; unde sequitur, si distantia inter puncta O & N infinitè diminuatur, parallelogrammum MEOD, (jam MRNS factum) minus fore quocunque

* Art. 182. dato; ac proinde* parallelogramma PQRM, PQNS, erunt & sibi invicem & spatio curvilineo PQMN æqualia, ideoque pro sibi invicem usurpari possunt.

2do. Sit arcus MN portio Curvæ AMB ad Diametrum ejus magis magisque accedentis; liquet demonstrationem eandem esse ac in primo casu, excepto quod parallelogrammum PQNS jam inscriptum evadat.

3tio. Detur Curva ABG ex pluribus constans portionibus, quarum aliæ, sicut AB, a Diametro AG continuo recedant, aliæ vero, sicut BG, ad eam magis magisque accedant. Dico fieri non posse, ut puncta, sicut B, has portiones separantia, super arcus MN cadant; si enim cadere possent, punctum B ad punctum M propius accederet, quam ipsum N; contra Hypothesin. Liquet ergo hunc casum in prioribus necessario comprehendi. Cor-

Corollarium I.

184. Hinc, si Ordinatim applicetur recta CB ipsi PM parallela, & si fingamus Curvæ portionem AB in arcus infinite parvos dividi, ad instar arcus MN; spatium ACB sub rectis AB, AC, & Curvæ portione AB comprehensum, æquabitur summæ parallelogrammorum omnium ipsi PQRM vel PQNS æqualium. Atque ita de omnibus Curvæ ABG partibus dicendum.

Corollarium II.

185. Sit CMDOC figura quæcunque inter parallelas CE, DF, comprehensa: & si inter has parallelas agantur duæ rectæ MO, NL, ipsis CE, DF, parallelæ, quarum distantia infinitè parva sit; dico spatium OMNL, in figura CMDOC sub rectis MO, NL contentum, æquari rectangulo MO × MR vel OS. Fig. 88.

Ductâ enim rectâ AB parallelis CE, DF perpendiculari, ipsis MO, NL etiam parallelis in punctis P & Q occurrente; liquet spatium PMNQ æquari rectangulo PMRQ, ita & spatium POLQ rectangulo POSQ; unde erit spatium OMNL æquale rectangulo OMRS vel OM × PQ.

Corollarium III.

186. Constat ex consectario præcedenti, si duæ quævis figuræ CMDOC, EGFHE inter parallelas CE, DF comprehensæ ejusmodi fuerint, ut ducta alicubi rectâ MH ipsis CE, DF, parallela, partes MO, GH, istiusce rectæ in figuris CMDOC, EGFHE, contentæ semper sint in ratione data; constat, inquam, binas has

figuras (spatia scilicet sub iis comprehensa) esse etiam ad invicem in ratione data; actâ enim recta NK ipsi MH parallela, ita ut distantia ejus ab MH infinite parva sit; ductâ etiam AB ipsis CE, DF, perpendiculari, parallelis autem MH, NK, in punctis P & Q occurrente;
* Art. 185. liquet, spatium OMNL *æquari rectangulo MO × PQ, ita & spatium GHKI æquari rectangulo GH × PQ. Hæc igitur spatia erunt ad invicem ut MO ad GH; & cum hoc verum sit, quacunque in figu-
* 12. El. 6. ræ parte ducatur recta MH, sequitur *summam spatiolorum omnium MNLO, hoc est, spatium CMDOC esse ad summam spatiolorum omnium GHKI, hoc est, ad spatium EGFHF, in ratione data ipsius MO ad GH.

Eodem ratiocinio probatur, figuræ CMDOC portionem aliquam MDO esse ad figuræ EFGHE portionem similem GFH in eadem ratione data: atque ita de partibus residuis CMO, EGH dicendum.

Manifestum est, si data ratio fuerit ratio æqualitatis, hoc est, si partes MO, GH ipsius rectæ MH semper æquentur, spatia CMDOC, EGFHE, ita & partes MDO, GFH, & CMO, EGH semper fore æqualia.

Lemma III.

Fig. 89. 187. Si *super Curvam quamvis sumatur arcus* MN *infinitè parvus; & per illius terminos agantur tangentes* MT, NT, *sibi mutuo occurrentes in puncto* T, *ducaturque subtensa* MN, *& recta* NS *ipsi* MT *productæ normalis; dico subtensam* MN, *vel summam tangentium* MT, NT, *vel denique rectam* MS *pro arcu* MN *usurpari posse.*

Necesse est omnis Curva vel tota concava sit versus unam

unam partem, vel ex pluribus portionibus conſtet, quarum aliæ verſus unam aliæ verſus alteram partem Concavæ fuerint; jam vero puncta, quibus hæ portiones ſeparantur, dari *non poſſunt ſuper arcus MN infinitè parvos: alioqui enim propiora eſſent puncto M quàm ipſum N, contra Hypotheſin; poni igitur ſemper poteſt arcum MN eſſe Curvæ portionem verſus unam partem concavam. * Art. 183. N. 3.

Jam ſi ſuper Curvam ex parte puncti N ſumatur arcus MO finitæ magnitudinis, aganturque ſubtenſa OM, tangens OG, & recta OD ipſi NS parallela, erit *1mo.* Ob triangulum MDO rectangulum ad D, tangens MD minor ſubtenſa MO, & multo minor arcu MNO; adeo ut arcus MNO (ita & ſubtenſa ejus MO) major ſit quàm MD, & minor quam tangentium ſumma MG, OG; *2do.* Ob arcum MNO verſus eandem partem concavum, liquet, ſi per punctum quodvis N arcus MO ducatur tangens TR, puncta T, R, ubi tangentibus MG, OG, occurrit, erunt inter puncta M, G, & O, G; ideoque angulus OGD triangulo TGR externus, major erit angulo RTG vel NTS.

His poſitis, ſi ducantur rectæ ME, MF, tangentibus OG, NT, parallelæ, ipſi DO in punctis E & F occurrentes; fingaturque punctum O ſuper Curvam verſus M moveri; liquet angulum OGD, vel EMD ipſi OGD æqualem, continuo diminui, adeo ut, puncto O ad M pervento, prorſus evaneſcat; ex eo quod tangens OG cum tangente MG coincidat: unde conſtat rectam ME continuo diminui, donec tandem ipſi MD æqualis evadat; ideoque puncto O in N exiſtente, recta ME tum in MF exiſtens differet a tangente MD quantitate minore quâvis datâ; ac proinde *rectæ TN, TS, ſibi * Art. 182.

R 2 mutuo

mutuo æquales erunt: unde tangentes MT, TN, si-simul sumptæ, æquales erunt & rectæ MS, & arcui MN, & subtensæ MN.

Corollarium I.

188. Cum angulus FMD, vel NTS ipsi FMD æqualis, infinitè exiguus sit, posito punctum N infinite propius ad punctum M accedere, erit in triangulo MTN angulus internus NMT externo NTS minor, infinitè parvus, hoc est, minor quovis dato; ideoque nulla recta per punctum M duci potest, quæ in angulo TMN cadat; unde patet, rectas MT, NM, sibi mutuo congruere, ac proinde tangens haberi potest pro recta, quæ per duo Curvæ puncta infinitè propinqua transeat.

Corollarium II.

189. Si ponamus Curvam quamvis in arcus MN infinitè parvos dividi, liquet si loco ipsorum arcuum sumantur eorum subtensæ, exinde generari Polygonum lateribus numero infinitis, singulis etiam infinitè exiguis, quod Polygonum pro ipsa Curva usurpari potest, ab ea enim non * omnino differet.

* Art. 187.

Porro exigua illa Polygoni latera ex utraque parte producta, erunt istiusce Curvæ tangentes, ex eo quod per duo ipsius puncta infinitè propinqua transeant.

Lemma IV.

Fig. 90. 190. Si *Circulum* AMG *tetigerit recta* AD, *secueritque utcunque recta* APG; *& si a quovis puncto* M *in Circuli peripheria ducatur recta* MP *tangenti* AD *parallela, compleaturque*

pleaturque parallelogrammum APMD; *dein accedat punctum* M *ad punctum* A; *dico chordam* AM, *&* *tangentem* AD, *vel* MP, *fore ultimo in ratione æqualitatis.*

Acta enim MG, ob similia triangula AMG, AMD, erit AM : AD :: AG : MG; coeuntibus autem punctis A & M, erit AG ad MG in ratione æqualitatis, adeoque erit etiam AM ad AD in ratione æqualitatis.

COROLLARIUM.

191. Unde cum sit universaliter AP : AM :: AM : AG, coeuntibus A & M, erit AP : MP :: MP : AG, adeoque erit in hoc casu $\overline{MP}^2 = AP \times AG$.

LEMMA V.

192. SI *figura Elliptica secundum longitudines Ordinatarum ad Diametrum quamvis pertinentium dilatetur vel coarctetur, producatur, vel contrahatur, vel si servatis Ordinatarum longitudinibus & intersectionibus cum Diametro, mutentur utcunque earum inclinationes; dico figuram novam inde genitam nihilominus Ellipticam fore, si modo mutationes omnes lineares in longitudinem ubique proportionales fuerint, & angulares æquales.*

Centro C, & diametris quibusvis conjugatis A*a*, D*d*, Fig. 91. utcunque ad invicem inclinatis describatur Ellipsis AMD*a* in qua sit MP Ordinatim applicata ad diametrum A*a*; dein manentibus semidiametri CD & Ordinatarum omnium MP positionibus, mutetur longitudo CD in longitudinem CE, & longitudines omnes PM in totidem PN, ita tamen, ut singulæ PN sint ad singulas PM ut CE ad.

CE ad CD; dico figuram novam ANE*a* Ellipsin esse, semidiametris conjugatis AC, CE, descriptam;

* Hyp. Nam * $PN : PM :: CE : CD$,

Unde $\overline{PN}^2 : \overline{PM}^2 :: \overline{CE}^2 : \overline{CD}^2$;

* Art. 48. Est autem * $\overline{PM}^2 : AP \times Pa :: \overline{CD}^2 : \overline{CA}^2$,

* 22. El. 5. Ergo * $\overline{PN}^2 : AP \times Pa :: \overline{CE}^2 : \overline{CA}^2$.

Coincidit itaque omni ex parte figura nova ANE*a* cum Ellipsi, semidiametris conjugatis AC, CE, descriptâ.

Fig. 92. Casus 2*dus*. Servatis jam Ordinatarum longitudinibus & intersectionibus cum diametro, mutetur utcunque angulus ACE in angulum ACF, & anguli omnes APN in totidem APO, ita tamen, ut Ordinatæ omnes PO, CF, parallelæ sint, & oritur figura nova AOF*a*, quæ etiam Ellipsis erit semidiametris AC, CF, descripta, si modo similes fiant mutationes in altera semi-Ellipsi A*da*, nam ob nihil mutatas Ordinatarum longitudines aut intersectiones, erit etiamnum $\overline{PO}^2 : AP \times Pa :: \overline{CF}^2 : \overline{CA}^2$.

Corollarium.

193. Figura quævis Elliptica augendo vel minuendo, inclinando vel reclinando diametros suas conjugatas in aliam quamcunque ejusdem generis figuram migrare potest.

PROPOSITIO I.

Theorema.

194. Ratio *parallelogrammi circa diametros quasvis conjugatas Ellipseos cujuscunque circumscripti ad figuram Ellipticam inscriptam eadem semper est & immutabilis.*

Nam

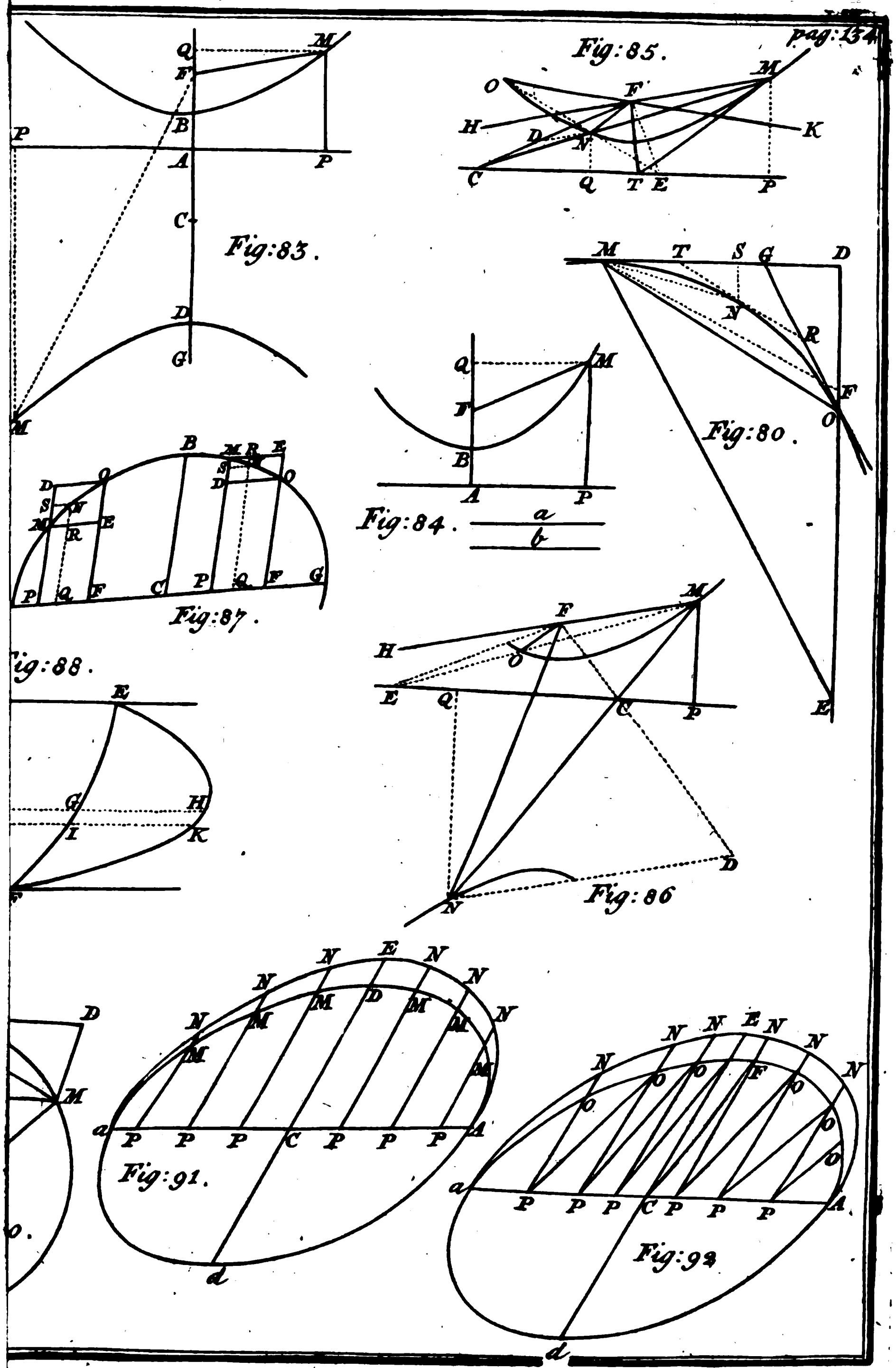
Fig: 85.
Fig: 83.
Fig: 80.
Fig: 84.
Fig: 87.
Fig: 88.
Fig: 86
Fig: 91.
Fig: 92

Nam si dilatando, coarctando, producendo, vel utcunque ordinando figuram inscriptam, augeatur illa vel minuatur in data aliqua ratione, augebitur vel minuetur in eadem ratione etiam figura circumscripta; sed figura quælibet Elliptica augendo vel minuendo, inclinando vel reclinando Diametros suas conjugatas in aliam qnamcunque ejusdem generis figuram * migrare potest; patet itaque * Art. 193. propositum.

COROLLARIUM I.

195. PARALLELOGRAMMA omnia circa ejusdem vel æqualium Ellipsium diametros quasvis conjugatas descripta æquantur inter se.

COROLLARIUM II.

196. ELLIPSEÔN areæ sunt ad invicem ut parallelogramma circa diametros conjugatas, adeoque ut rectangula sub axibus.

COROLLARIUM III.

197. IMMUTABILIS illa ratio inter Parallelogramma & figuras Ellipticas inscriptas, est ea ipsa, qua quadrati circulo circumscripti ad inscriptum circulum, seu quam 6. Archim. habet 14 ad 11 quam proximè; & sic reducitur Quadratura Ellipseos ad circuli quadraturam, nam circulum Theor. considerare licet tanquam Ellipsin cujus diametri quævis conjugatæ æquantur inter se, & ad rectos sunt angulos.

PRO-

PROPOSITIO II.

THEOREMA.

Fig. 27. 198. SI *per Ellipseos vel Hyperbolæ* AM *punctum quodvis* A *ducatur diameter* Aa, *cujus parameter* p, *& Ordinata* MP, *& si, manentibus diametri & Ordinatarum positionibus terminoque* A *& parametro* p, *abire intelligatur terminus alter* a *ad distantiam infinitam; dico figuram illam* AM *ultimo transformatam iri in Parabolam Apolloniam cujus diameter est* AP, *Parameter ad Diametrum* p, *& Ordinatæ singulæ ipsi* PM *parallelæ.*

* Art. 49. Nam ex natura Ellipseos * erit

$\overline{MP}^2 : AP \times Pa :: p : Aa$,

Sed $p : Aa :: AP \times p : AP \times Aa$,

Unde $\overline{MP}^2 : AP \times p :: AP \times Pa : AP \times Aa :: Pa : Aa$.

Abeat jam punctum a ad distantiam infinitam, & ratio Pa ad Aa fiet ultimo ratio æqualitatis, ergo in hoc casu

* Art. 6. erit $\overline{MP}^2 = AP \times p$, notissima * Proprietas Parabolæ cujus Diameter est AP, Parameter ad Diametrum p, & Ordinata MP.

SCHOLIUM.

199. HINC proprietates omnes Parabolæ, ex similibus vel Ellipseos vel Hyperbolæ proprietatibus derivari possunt; quorum exemplum unum aut alterum apponam.

Fig. 27. Ex. 1*mum*. Parabola unico tantum gaudet umbilico F, ob recessum alterius f una cum termino a in infinitum,

* Def. 9. II. Ex. 2*dum*. Diametri omnes Ellipseos * concurrunt in centro

centro C; ergo Diametri omnes Parabolæ ſunt ſibi invicem parallelæ, utpote non concurrentes, niſi ad diſtantiam infinitam.

3*tium.* Si Ellipſeos vel Hyperbolæ Diametro A*a* conjugata ſit B*b*, * erit $AF \times Af = \overline{BC}^2$; adeoque $4AF \times Af = \overline{Bb}^2$, ſed ex natura Parametri *p*, quæ pertinet ad Diametrum A*a*, * erit $Aa : Bb : Bb ; p$; adeoque $Aa \times p = \overline{Bb}^2$. Fig. 12, & 13. * Art. 42. * Def. 8. II. & III.

Ergo $4AF \times Af = Aa \times p$.

Et * $p : 4AF :: Af$ ſeu $Fa : Aa$. *16. El. 6.

Abeat jam terminus *a* ad diſtantiam infinitam, & erit A*f* ad A*a* ultimo in ratione æqualitatis; ergo in Parabola $p = 4AF$, idque ſive AP ſit Axis, ſive alia quævis Diameter.

4*tum.* Producatur Axis A*a* ultra terminum A ad E, ita ut æquales fiant AE, AF; centroque *f* & intervallo *f*E deſcribatur arcus circularis ED, & per M educatur recta *f*MD, & erit $fD\ (= fE = Aa) = Mf + MF$; ſubducatur utrinque M*f*, & erit $MD = MF$; abeat jam punctum *a* una cum puncto *f* ad infinitum, & arcus finitus ED (abeunte centro ad infinitum) jam expandetur in recta axi A*a* & rectæ MD perpendiculari; adeoque ſi in axe Parabolæ producto capiatur $AE = AF$, & per E ducatur ED ipſi AE perpendicularis, & ſi ex quovis puncto M ducatur tum MF ad umbilicum F, tum MD axi parallela, rectæ MF, MD ſemper æquales erunt inter ſe. Fig. 93.

5*um.* Tangat recta TMS Ellipſin in puncto M, occurratque ſemidiametro CA productæ in T, & * erit, Fig. 27. * Art. 65.

CP : CA :: CA : CT, unde CP : PA :: CA : AT,

Et viciſſim, CP : CA :: PA : AT.

Abeat centrum C in infinitum, & erit CP ad CA ultimo in ratione æqualitatis; ergo in Parabola erit $AP = AT$.

Fig. 24. *6tum.* Esto tangenti MT perpendicularis MG, occurrens Axi Aa in G; & *erit $CP : PG :: Aa : p$.
* Art. 69.
Et vicissim, $CP : Aa :: PG : p$.

Abeat jam centrum C ad infinitum, & erit $CP = \frac{1}{2} Aa$; ergo in Parabola erit $PG = \frac{1}{2} p$.

Fig. 27. *7mum.* Per umbilicos F, f, ducantur MF, Mf, & hæ cum tangente æquales angulos TMF, SMf efficiunt: capiatur in recta Mf longitudo quævis finita MO; dein abeat punctum f ad infinitum, & recta MO erit ultimo axi Aa parallela; ergo in Parabola recta MF per umbilicum manentem traducta, & diameter MO æquales cum tangente angulos conficiunt.

PROPOSITIO III.

Theorema.

Fig. 94. 200. Circulus, *qui tangit Sectionem Conicam, abscinditque ex Diametro* Aa *per contactum transeunte longitudinem* AG *æqualem Parametro* p *ad Diametrum illam pertinenti, curvaturam habet Sectionis in loco contactus. Et vice versâ.*

Non enim: sed, si fieri potest, coeat alius circulus cujus Diameter AK, cum Sectione Conica AM in loco A, secetque Diametrum Aa, cujus Parameter p, in puncto K, quæcunque tandem fuerit longitudo AK, & ad Diametrum Aa duc Ordinatam MP; & si sectio Ellipsis fuerit vel Hyperbola, * erit,
* Art. 49. & 89.

$$\overline{MP}^2 : AP \times Pa :: p : Aa :: AP \times p : AP \times Aa,$$

Unde $\overline{MP}^2 : AP \times p :: AP \times Pa : AP \times Aa :: Pa : Aa$.

Coeat

Coeat jam punctum P cum puncto A, & erit Pa ad Aa ultimo in ratione æqualitatis, ergo in hoc casu erit $\overline{MP}^2 = AP \times p$, ut * universaliter in Parabola; sed lineola evanescens PM jam communis est Ordinata tum Sectioni tum circulo æquicurvo, cujus Diameter AK, propter suppositam Curvarum coincidentiam, ergo est etiam $\overline{MP}^2 = AP \times AK = AP \times p$, & $AK = p =$ * AG; Quod est absurdum.

* Art. 7.

* Hyp.

SCHOLIUM.

201. Hinc facilè describi potest Circulus, qui curvaturam habeat Sectionis in puncto quovis dato; revocatur enim problema ad hoc nempe circulum ducere, qui datam positione rectam in dato puncto tangat, & per aliud datum punctum transeat.

LEMMA VI.

202. SI *Ellipsin vel Hyperbolam quamvis datam tangat indefinita recta* QPT, *& ab umbilicis* S *&* F *demittantur ad tangentem perpendiculares* SQ, FT, *dico rectangulum sub his perpendicularibus æquari quadrato axis semiconjugati.* Fig. 95. 96. 97.

Scilicet $SQ \times FT = \overline{OD}^2$.

Ducantur SP, FP, productaque SP ipsi FT productæ occurrat in C, & recta TO, quæ bisecat FS in centro O propter æquales angulos TPF, TPC, bisecabit * etiam FC in T, & proinde parallela erit ipsi SC, & æqualis ejus dimidio, vel dimidio summæ aut differentiæ rectarum SP, PF; * vel semiaxi principali OA vel OB. Producta itaque TO, donec occurrat ipsi QS (productæ) in t; circulus centro O descriptus transibit # per puncta B, T, t,

* 2. El. 6.

* Art. 39. 80.

#(ob æquales OB, OA, OT, Ot, et diametrum Tt recto angulo t QT substensam)

* 35. El. 3. A, Q; adeoque * rectangulum tSQ vel $FT \times SQ$ erit ubi-
* Art. 42. & 83. que æquale rectangulo dato ASB, *hoc est, quadrato $\overline{OD}^2$.

Corollarium I.

203. Perpendiculum SQ in data figura erit ut $\sqrt{\frac{SP}{FP}}$. Est enim (ob sim. triang.) $SP : SQ :: FP : FT$. Ergo $\frac{SQ \times FP}{SP} = FT$, & $\frac{\overline{SQ}^2 \times FP}{SP} = SQ \times FT$; datur itaque quantitas $\frac{\overline{SQ}^2 \times FP}{SP}$, adeoque $\overline{SQ}^2$ erit reciprocè ut $\frac{FP}{SP}$, vel directè ut $\frac{SP}{FP}$; & SQ est ut $\sqrt{\frac{SP}{FP}}$.

Corollarium II.

204. Cum sit $SQ \times FT = \overline{OD}^2$, longitudines SQ, OD, FT erunt continuè proportionales, & $\overline{SQ}^2$ erit ad $\overline{OD}^2$ ut SQ ad FT, vel ut SP ad PF; unde erit $\overline{SQ}^2 = \frac{\overline{OD}^2 \times SP}{PF}$.

Corollarium III.

Fig. 98. 205. Manentibus S & A punctis, recedat umbilicus F ad infinitum, & longitudines SB & PF infinitæ erunt
* Art. 182. cum differentia finita, ergo SB * erit ad PF in ratione æqualitatis, unde in Parabola est $\overline{SQ}^2$ vel $\frac{ASB \times SP}{PF} = AS \times SP$, ideoque SA, SQ, SP erunt continue proportionales.

PRO-

PROPOSITIO IV.

Theorema.

206. Rectangulum *sub distantiis* SP, PH, *puncti cujusvis* P *ab utroque Ellipseos foco, æquatur quadrato semi-diametri* CD *conjugatæ cum ea quæ per punctum illud ducitur.* Fig. 99.

Scilicet $SP \times PH = \overline{CD}^2$.

Ductâ * enim per punctum P tangente ZPY, demissisque ab umbilicis S, H, & a centro C ad eam perpendicularibus SY, HZ, CR; similia erunt triangula SPYS, HPZ, adeoque erit * Art. 199. (ob angulos aequales YPS, HPZ)

$SP : SY :: HP : HZ :: SP + PH : SY + HZ :: 2AC : 2CR$.

Sed $2AC : 2CR :: AC : CR ::$ * $CD : CB$. * Art. 67. & 145.

Unde $\overline{SP}^2 : \overline{SY}^2 :: \overline{CD}^2 : \overline{CB}^2$.

Est autem $\overline{SP}^2 : \overline{SY}^2 :: SP \times PH : SY \times HZ$.

Ergo $SP \times PH : SY \times HZ :: \overline{CD}^2 : \overline{CB}^2$.

Sed * $SY \times HZ = \overline{CB}^2$; ideoque $SP \times PH = \overline{CD}^2$. * Art. 202.

PROPOSITIO V.

Theorema.

207. Dato *umbilico Sectionem Conicam describere, quæ datam positione rectam* RP *in dato puncto* P *contingat, quæque datam habeat curvaturam in loco contactûs.* Fig. 99.

Esto Ellipseos Umbilicus datus S; agatur SP secans tum Diametrum DK in E, tum Ordinatim applicatam *Qv* in *x*. Patet EP æqualem esse semi-axi majori AC,

eo

eo quod acta ab altero Ellipseos umbilico H linea HI ipsi EC parallela (ob æquales CS, CH) æquentur ES, EI, adeo ut EP semi-summa sit ipsarum PS, PI, id est (ob parallelas HI, PR, & angulos *æquales IPR, HPZ) ipsarum PS, PH, quæ conjunctim axem totum 2AC adæquant.

* Art. 72.

Ob datum angulum RPS dabitur huic æqualis angulus ZPH, & proinde dabitur positione recta PH, quæ per alterum umbilicum H transit. In recta PF, vel in eadem (si opus) producta, capiatur longitudo PO æqualis Diametro Curvaturæ datæ in P, quâ quidem Diametro describatur circulus PQ secans tum Diametrum Sectionis GP in V, tum rectam PS in L, & arcus quam minimus PQ æque pertinebit ad circulum jam descriptum ac ad Sectionem jam describendam; porro ex natura circuli, longitudo $PV = \frac{\overline{Qv}^2}{Pv}$ & longitudo $PL = \frac{\overline{Qx}^2}{Px}$ ergo ob æquales $\overline{Qv}^2$, $\overline{Qx}^2$, erit $PV : PL$ (:: $\frac{1}{Pv} : \frac{1}{Px}$ :: $Px : Pv$ :: $PE : PC$ :: $AC : PC$; vel) $SP + PH : 2PC$; ergo $\overline{SP + PH} \times PL = 2PC \times PV$: ostensum *est autem longitudinem PV æqualem esse Parametro ad Diametrum $PG = \frac{4\overline{CD}^2}{2PC}$; ergo $2PC \times PV = 4\overline{CD}^2$ $*= 4SP \times PH$; ergo $\overline{SP + PH} \times PL = 4PS \times PH$. Unde prodit $PH = \frac{SP \times PL}{4SP - PL}$; dantur autem & SP & PL; ergo PH & positione & longitudine; & Sectio Conica umbilicis S, H, per punctum P descripta tanget rectam RP in

* Art. 200.

* Art. 206.

in loco P, & in hoc curvaturam circuli Diametri PO descripti habebit. Q. E. I.

Sectio Ellipsis erit, vel Parabola, vel Hyperbola, prout longitudo PL minor fuerit, vel æqualis, vel major quam 4PS.

PROPOSITIO VI.

THEOREMA.

208. SI *in Sectione Conica ducantur duæ quævis parallelæ* BD, EF, *ad Sectionem terminatæ; junganturque earum termini duabus rectis* BE, DF; *dico segmenta* BMEB, DMFD, *sub Sectionis portionibus, & rectis parallelarum terminos jungentibus comprehensa, sibi mutuo æquari.* Fig. 100. & 101.

Productis enim subtensis BE, DF, usque dum in puncto aliquo G concurrant, ductâque per G & per medium ipsius BD rectâ GH, constat rectam EF ipsi BD parallelam ab ipsâ GH bifariam secari in puncto K; ita & rectam OO in P; erit igitur recta HK Sectionis *Diameter, cujus Ordinatæ ex utraque parte erunt parallelæ BD, EF; ideoque si in Sectione per punctum quodvis P ducatur recta MM ipsis BD, EF, parallela, ea Sectioni *occurret in duobus punctis M & M, a puncto P æquè remotis: unde constat partes MO, OM ejusdem rectæ MM ipsi BD parallelæ, inter segmenta BMEB, DMFD contentas, sibi mutuo æquales esse, quacunque in parte inter rectas BD, EF, cadat ipsa MM; segmenta *igitur BMEB, DMFD, æquantur.

* Art. 153. * Art. 151. * Art. 186.

COROLLARIUM I.

209. QUONIAM MP ipsi PM semper sit æqualis, sequitur, Fig. 100.

1mo. Trapezia Conica KHBE, KHDF, ſibi invicem æquari.

2do. Quando recta BD Sectionem in puncto A tangit, triangula Conica AKE, AKF æqualia eſſe; ideoque etiam ſegmenta AEMA, AFMA; ex eo quod triangulum AEF in duas partes æquales a recta AK per medium ipſius EF tranſeunte dividatur.

Corollarium II.

Fig. 100. 210. Si, Sectione exiſtente Parabola, Ellipſi, vel Hyperbola, parallelarum BD, EF, termini rectis BF, DE, ſe invicem decuſſantibus jungantur, erunt ſegmenta BFDAB, DEBAD, inter ſe æqualia, triangula enim BFD, BED, inter eaſdem parallelas BD, EF, & ſuper eadem baſi erunt æqualia: unde ſi ex alterâ parte addatur ſegmenmentum DMFD + BADB, & ex alterâ BMEB + BADB, erunt tota BFDAB, & DEBAD ſibi mutuo æqualia; nam ſegmentum DMFD ipſi BMEB æquatur.

PROPOSITIO VII.

Theorema.

Fig. 102 103 & 104. 211. Si *in Ellipſi, Hyperbola, vel Hyperbolis oppoſitis, agantur duæ rectæ* BD, EF *ſibi mutuo parallelæ, & ad Sectionem terminatæ; ut & è centro* C *ſemidiametri* CB, CE, CD, CF, *erunt Sectores Elliptici vel Hyperbolici* CBE, CDF, *ſibi mutuo æquales.*

Ducta enim per media H, K, ipſarum BD, EF, diametro CK, triangula CHB, CHD, ita & CKE, CKF, erunt æqualia, utpote quæ communem verticem C, baſes autem HB,

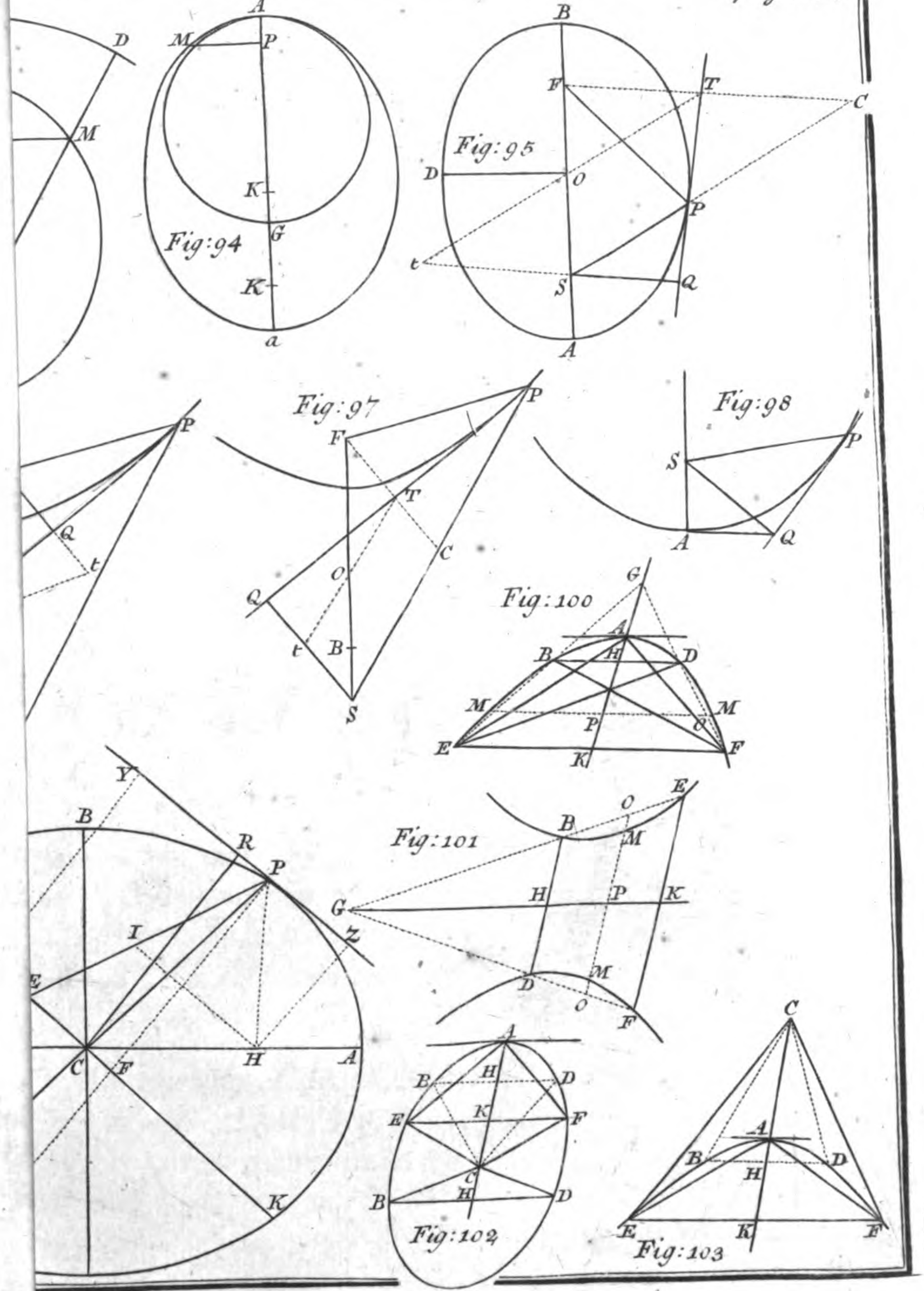

Fig: 94

Fig: 95

Fig: 97

Fig: 98

Fig: 100

Fig: 101

Fig: 102

Fig: 103

HB, HD; & KE, KF, æquales habeant; ideoque (in Fig. 103.) erit KHBE + CBE = CKE — CHB = CKF — CHD = KHDF + CDF, Porro in Fig. 102. 104. KHBE — CBE = ± CHB ∓ CKE = ± CHD ∓ CKF = KHDF — CDF, ſed trapezia Conica KHBE, KHDF, * æquantur: unde ſectores Elliptici vel Hyperbolici CBE, CDF erunt æquales. * Art. 209.

Corollarium I.

212. Si in Ellipſi, vel Hyperbola, recta BD ipſi EF parallela, fieret tangens in puncto A; liquet ſectores CAE, CAF, eſſe inter ſe æquales: productâ enim ſemidiametro CA, uſque dum rectæ EF in puncto K occurrat, illa recta bifariam ſecabitur in illo puncto; ac proinde triangula rectilinea CKE, CKF, æqualia erunt: ſed * triangula Conica AKE, AKF, æquantur; ergo ſectores CAE, CAF, ſunt inter ſe æquales. Fig. 102. 103. * Art. 209.

Corollarium II.

213. Hinc facile biſecatur ſector quivis Ellipticus vel Hyperbolicus CEF; ducatur ſcilicet ſemidiameter CA, illius ſectoris ſubtenſam EF in puncto K bifariam ſecans.

PROPOSITIO VIII.

Theorema.

214. Si *in ſemicirculo* ADH, *cujus Diameter fuerit Ellipſeos axis major* AH, *agatur per ipſius* AH *punctum quodvis* P *recta* PM *axi normalis, Ellipſi in puncto* M, *circulo autem in puncto* N *occurrens*; *&* *per puncta* M, N, Fig. 105.

T *ducan-*

ducantur ad centrum C *rectæ* CM, CN; *dico sectorem Ellipticum* CAM *esse ad sectorem circularem* CAN, *ut* CB *dimidium axis minoris, ad* CA *vel* CD, *dimidium axis majoris.*

Scilicet CAM : CAN :: CB : CA vel CD.

* Art. 50. 63. Nam * $\overline{PM}^2 : \overline{CB}^2 :: AP \times PH : AC \times CH$ vel CA.

* 13. El. 6. Et * $\overline{PN}^2 : \overline{CD}^2 :: AP \times PH : AC \times CH$ vel $\overline{CA}^2$.

Ergo $\overline{PM}^2 : \overline{CB}^2 :: \overline{PN}^2 : \overline{CD}^2$.

Vel PM : PN :: CB : CD. Et cum hoc idem eveniat, quacunque in parte cadat perpendicularis MPN, sequi-
* Art. 186. tur * spatium totum Ellipticum ABHA esse ad semicirculum ADHA, ita & illius spatii partem APM esse ad semicirculi partem APN, in ratione ipsius CB ad CD vel
* 1. El. 6. CA. Sed triangulum * CPM : CPN :: PM : PN.

Et PM : PN :: CB : CD vel CA. Unde erit APM $\mp$ CPM : APN $\mp$ CPN :: CB : CD vel CA. (signo existente + quando AP minor est quàm CA, & $-$ quando AP major est quàm CA) ideoque erit sector Ellipticus CAM ad sectorem circularem CAN ut CB ad CD vel CA;

Corollarium I.

215. Hinc sector Ellipticus CAM est ad sectorem circularem CAN; sicut tota Ellipsis ad totum circulum; nam semiellipsis est ad semicirculum in ratione ipsius CB CD vel CA.

Corollarium II.

216. Sector Ellipticus CAM æquatur rectangulo sub arcu AN, & dimidio radii CB; nam area totius circuli

culi * æquatur triangulo cujus basis est peripheria circuli, altitudo autem semidiameter; unde circuli pars aliqua CAN æqualis est triangulo, cujus basis sit arcus AN, altitudo autem CD; vel * rectangulo sub arcu AN & $\frac{1}{2}$CB. * 5. Theor. ex Archim. * 41. El. 1.

Sed * CAM : CAN :: $\frac{1}{2}$CB : $\frac{1}{2}$CD, * Art. 209.

Et * $\frac{1}{2}$CB : $\frac{1}{2}$CD :: AN $\times \frac{1}{2}$CB : AN $\times \frac{1}{2}$CD. * 1. El. 6.

Ergo CAM : CAN :: AN $\times \frac{1}{2}$CB : AN $\times \frac{1}{2}$CD.

Unde erit CAM $=$ AN $\times \frac{1}{2}$CB.

Corollarium III.

17. Hinc si in axe majori AH per punctum quodvis G a puncto P diversum, agatur ad axem normalis GE, Ellipsi in puncto E, circulo autem in puncto F occurrens; erit sector ACE : ACM :: ACF : ACN.

Nam * ACE : ACF :: CB : CD :: ACM : ACN, * Art. 214.

Ergo ACM : ACE :: ACN : ACF.

Unde facile invenitur sector Ellipticus ACM, qui sit ad sectorem Ellipticum ACE in ratione data. Inveniatur scilicet sector circularis ACN, qui sit ad sectorem ACF, in ratione data; vel (quod eodem recidit) dividatur arcus ANF vel angulus ACF in ratione data.

PROPOSITIO IX.

Theorema.

218. Si *in duabus Hyperbolis* AM, AN, *vel* BM, DN, *quarum centrum* C, *semidiameter communis recta* AC, *diametri autem conjugatæ* CB, CD, *per punctum quodvis* P *semidiametri* CA (*productæ, si opus*) *ducatur recta* PM *ipsi* CD *parallela, Hyperbolis occurrens in punctis* M *&* N, *&* Fig. 106. & 107.

 jungantur

jungantur MC, NC; *dico sectores Hyperbolicos* CAM, CAN; *vel* CBM, CDN, *esse ad invicem in ratione semidiametrorum conjugatarum* CB, CD.

* Art. 88, & 126. Nam * $\overline{PM}^2 : \overline{CB}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CA}^2$,

Et $\overline{PN}^2 : \overline{CD}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CA}^2$,

Unde $\overline{PM}^2 : \overline{CB}^2 :: \overline{PN}^2 : \overline{CD}^2$.

Et $PM : PN :: CB : CD$.

Et cum hoc idem eveniat, quacunque in parte cadat pa-
* Art. 186. rallela PMN, liquet * spatia Hyperbolica APM, APN, vel (in Fig. 107.) CPMB, CPND, esse inter se, ut CB
* 1. El. 6. ad CD. Sed triangula CPM, CPN, * sunt ad invicem, ut PM ad PN; vel CB ad CD.

* Fig. 106. Unde * $CPM - APM : CPN - APN :: CB : CD$.

Hoc est, $CAM : CAN :: CB : CD$.

Et in Fig. 107. erit

$CPMB - CPM : CPND - CPN :: CB : CD$.

Scilicet $CBM : CDN :: CB : CD$.

Corollarium.

219. Si duæ semidiametri conjugatæ CA, CD, fue-
* Def. 17. Lib. 3. rint æquales, Hyperbola AN vel DN erit * æquilatera; atque ita inventa quadratura sectorum Hyperbolicorum CAN, vel CDN, haberetur etiam quadratura sectorum CAM vel CBM, quorum bases sunt portiones AM vel alterius Hyperbolæ, positâ semidiametro conjugatâ CB magnitudinis cujuscunque; nam ratio sectorum CAM, CAN, vel CDN, CBM, per rectas CD, CB, designata, datur. Unde liquet, datâ Hyperbolæ æquilateræ quadraturâ, dari etiam aliarum omnium Hyperbolarum quadraturam;

draturam; eodem modo, quo datâ circuli quadratura, daretur * etiam omnium quadratura Ellipsium. * Art. 214.

PROPOSITIO X.

THEOREMA.

220. SI *super Hyperbolæ* EBDF *asymptoton* CN *sumantur duæ partes* CK, CL, *quæ sint ad invicem, ut ejusdem Asymptoti duæ aliæ partes* CG, CH; *ducantur autem rectæ* GF, HD, KB, LE, *alteri Asymptoto* CP *parallelæ, Hyperbolæ etiam in punctis* F, D, B, E, *occurrentes; ut & semidiametri* CF, CD, CB, CE; *dico sectores Hyperbolicos* CBE, CDF, *sibi invicem æquari.* Fig. 108.

Agantur enim rectæ BD, EF, Asymptotis in punctis M, O, N, P, occurrentes, & ob parallelas KB, HD, erit

MB : MK :: DO : CH.

Et ob parallelas LE, GF, erit etiam

NE : NL :: FP : CG.

Sed *MB = DO. Et NE = FP; unde erit MK = CH; * Art. 103.
& NL = CG.

Jam vero *CG : CH :: CK : CL, * Hyp.

Hoc est, NL : MK :: CK : CL :: *LE : KB; * Art. 108.

Ergo LN : LE :: MK : KB.

Rectæ igitur NE, MB, hoc est, EF, BD, erunt parallelæ: ac proinde sectores Hyperbolici CBE, CDF, erunt *æquales. * Art. 211.

COROLLARIUM I.

221. SI partes CK, CL, Asymptoti CN sint ad invicem in ratione partium CS, CT, Asymptoti CP; ducantur

cantur autem KB, LE, Aſymptoto CP, ut & SD, TF, Aſymptoto CN parallelæ, conſtat, ſectores Hyperbolicos CDF, CBE, etiam æquari; actis enim Aſymptoto CP parallelis FG, DH; erit *

* Art. 108.

CG : CH : : HD vel CS : GF vel CT : : CK : CL,

Ergo CG : CH : : CK : CL. Unde * erunt ſectores CDF, CBE, ſibi mutuo æquales.

* Art. 220.

Corollarium II.

222. Si ſuper eadem Aſymptoto ſumatur pars CK ipſis CG, CH, tertia proportionalis; eadem ratione, qua in Theoremate, oſtendetur, rectam BF tangenti per punctum D ductæ eſſe parallelam; ideoque * erunt ſectores CDF, CDB, inter ſe æquales. Unde ſi ſuper Aſymptoto CL ſumantur partes quotcunque CG, CH, CK, CL, *&c.* in continua progreſſione Geometrica, & ex punctis G, H, K, L, *&c.* agantur rectæ GF, HD, KB, LE, *&c.* alteri Aſymptoto parallelæ, ſectores Hyperbolici CFD, CDB, CBE, *&c.* erunt omnes æquales inter ſe.

* Art. 212.

Et viciſſim, ſi ſectores Hyperbolici CFD, CDB, CBE, *&c.* fuerint inter ſe æquales, erunt rectæ CG, CH, CL, *&c.* in continua progreſſione Geometrica.

Corollarium III.

223. Hinc ſi CH prima ſit duarum mediarum proportionalium inter CG, CL, ſuper Aſymptoton CN; & agantur rectæ GF, HD, LE, alteri Aſymptoto CP parallelæ; ſector CDF erit ad ſectorem CFE ut 1 ad 3. Ita etiam, ſi CH prima ſit trium mediarum proportionalium inter CG, CL, erit ſector CDF ad ſectorem CFE, ut

ut 1 ad 4. Et universaliter, si litera *m* numerum quemvis integrum denotet, & sit GH prima tot mediarum proportionalium inter CG, CL, quot numerus *m* — *i* contineat unitates; erit sector CDF ad sectorem CFE ut *i* ad *m*.

SCHOLIUM.

224. Hinc Logarithmorum natura satis acuratè explicare potest.

Ponamus enim rectam CG unitatem denotare, ipsam autem CL ipsius CG decuplam esse, vel 10; dividaturque sector Hyperbolicus CFE in 10000000000 partes æquales. Jam si construatur tabula & in duas columnas dividatur, quarum altera numeros naturales, 1, 2, 3, 4, 5, 6, *&c.* Ordinatim dispositas contineat; altera autem ex adverso numeros artificiales, ea lege compositas, ut si CH numerum quemvis naturalem denotet, numerus artificiosus e regione collocatus designet, quot partes Sector Hyperbolicus CDF contineat præ numero partium in sectore CFE contentarum.

Numeri artificiales vocantur Logarithmi numerorum naturalium, quibus respondent. His positis.

1*mo*. Sint CH, CK, duo numeri naturales in se invicem ducendi. Jam si in Tabula inveniantur ipsorum CH, CK, Logarithmi, (sectores CFE, CTB, denotantes) addanque sibi mutuo; habebitur Logarithmus sectorem CFE designans, cujus è regione collocabitur numerus naturalis CL, factus ex duobus numeris CH, CK, in se mutuo ductis.

2*do*. Si numerus CL per numerum CK dividendus sit, subducatur ex Logarithmo CFE dividendi CL, Logarithmus

garithmus CFB divisoris CK, & residuum erit quoti CH Logarithmus BE vel CFD.

30. SI Radix quæcunque numeri CL fuerit extrahenda, ex gr. Cubica; dividatur in tres æquales partes Logarithmus numeri CL, unde habebitur CFD, Logarithmus numeri CH, radicis quæsitæ.

Omnia hæc constant ex eo quod sectores Hyperbolici
* Art. 220. CFD, CBE, tum sibi invicem * æquantur, cum est CG : CH :: CK : CL; & quod sectores CFD, CDB, CBE, tum sibi invicem æquantur, cum est CG : CH :: CH : CK :: CK : CL, &c.

PROPOSITIO XI.

THEOREMA.

Fig. 109. 225. SI *in utraque Asymptoto sumantur partes* CG, CL, *ita &* CR, CS, *ejusmodi, ut sint* $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$; *aganturque rectæ* GF, LE, RT, SV, *Asymptotis parallelæ; dico sectorem* CFE *esse ad sectorem* CTV, *ut* m *ad* n. *Literæ* m *&* n *numeros quosvis integros designant.*

Fiat enim $\sqrt[m]{CG} : \sqrt[m]{CL} :: CG : CH$.

Ita & $\sqrt[n]{CR} : \sqrt[n]{CS} :: CR : CQ$.

Ducanturque rectæ HD, QN, Asymptotis parallelæ; &
* Art. 221. erunt sectores Hyperbolici CFD, CTN, inter se * æ-
* Hyp. quales, ex eo quod sit * CG : CH :: CR : CQ. Jam vero ex natura progressionis Geometricæ recta CH erit prima tot mediarum proportionalium inter CG & CL, quot numerus $m - 1$ contineat unitates; ob eandem rationem

erit

erit recta CQ prima tot mediarum proportionalium inter CR & CS quot numerus $n-i$ contineat unitates,

Unde * CFE : CFD :: $m : i$, * Art. 223.

Et CTN vel CFD : CTV :: $i : n$,

Ergo * CFE est ad CTV in ratione composita ex m ad i & ex i ad n, hoc est ut m ad n. * Art. 130.

COROLLARIUM.

226. HINC dato sectore Hyperbolico CFE, ita & Hyperbolæ puncto quovis T, facile inveniri potest in eadem Hyperbola punctum aliud V ejusmodi, ut sit sector CFE ad sectorem CTV ut m ad n; sumatur scilicet CS, ita ut sit $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$, vel quod eodem recidit, $\sqrt[m-n]{CG} : \sqrt[m-n]{CL} :: CR : CS$.

Hoc est $CS = CR \times \sqrt[m-n]{\frac{CL}{CG}}^{-n}$.

PROPOSITIO XII.

THEOREMA.

227. SI *per terminos* B, F, *sectoris cujusvis Hyperbolici* CBF, *agantur rectæ* BK, FG, *alteri Asymptoto* CS *parallelæ & ad alteram* CL *terminatæ*; *dico sectorem Hyperbolicum* CBF *æquari spatio Hyperbolico* BKGF, *sub rectis* BK, FG, *uni Asymptoto* CS *parallelis, sub alterius Asymptoti* CL *parte* GK, *& sub Hyperbolæ denique portione* BF *comprehenso*. Fig. 110.

Scilicet CBF = BKGF.

Triangula enim CKB, CGF * æquantur: sublato com- * Art. 107.

muni triangulo CGA (rectæ CB, GF, se intersecant in puncto A) & residua BKGA, CAF, erunt æqualia; quibus addatur idem spatium Hyperbolicum BAF, & erit spatium BKGF sectori CBF æquale.

Corollarium I.

228. Ductis rectis BQ, FO, Asymptoto CL parallelis, & ad Asymptoton CS terminatis, eodem modo ostendi posset sectorem Hyperbolicum CBF æqualem esse spatio Hyperbolico BQOF; unde constat, spatia vel trapezia Hyperbolica BKGF, BQOF, sibi invicem æquari.

Corollarium II.

229. Unde manifestum est quæcunque in articulis 220, 221, 222, 223, 224, 225, & 226, de sectoribus Hyperbolicis fuerint demonstrata, æqualiter obtinere apud hæc trapezia, sectoribus ipsis æqualia.

PROPOSITIO XIII.

Theorema.

Fig. 121. 230. Sint BMF, HND, *duæ Hyperbolæ inter se diversæ, quarum eædem Asymptoti* CL, CS; *aganturque per duo quævis Asymptoti unius* CL *puncta* G, K, *rectæ* GEF, KHB, *alteri* CS *parallelæ; dico spatium Hyperbolicum* HKGD *esse ad spatium Hyperbolicum* BKGF; *sicut dignitas Hyperbolæ* HND *ad dignitatem Hyperbolæ* BMF.

Ducta enim, per punctum quodvis P portionis GK, rectâ PNM ipsis GD, KH parallela, Hyperbolæ BMF occurrente in M, Hyperbolæ autem HND in N; dicatur dignitas Hyperbolæ HND, aa; Hyperbolæ autem

* Art. 109. BMF, bb; sit CP $= x$; & * erit PN $= \frac{aa}{x}$, & PM $= \frac{bb}{x}$;

unde

unde PN : PM :: *aa* : *bb*. Et cum hoc verum ſit, quacunque in parte ipſius GK cadat punctum P, liquet *ſpatium Hyperbolicum HKGD : BKGF :: *aa* : *bb*. * Art. 186.

Corollarium.

231. Quando dignitates Hyperbolarum HND, BMF, ſunt ad invicem ut numerus *m* ad numerum *n*, ſemper inveniri poteſt in Hyperbola HND trapezium Hyperbolicum RSVT æquale trapezio Hyperbolico GKBF in altera Hyperbola BMF, datis rectis CG, CK, CR; liquet *enim trapezium GKHD eſſe ad trapezium GKBF, ut *m* & *n*; ideoque in eadem Hyperbola HND inveniendum eſt trapezium RSVT, quod ſit ad trapezium GKHD ut *n* ad *m*; id autem fit * capiendo CS ea lege, ut ſit $\sqrt[m-n]{CG} : \sqrt[m-n]{CK} :: CR : CS$.

* Art. 225. 228.

* Art. 225. 229.

DEFINITIONES.

IV.

Sit AC recta indefinita, cujus terminus fuerit punctum A; ſit AMB curva ejuſmodi, ut, ſi per punctum quodvis P agatur recta MP, quæ cum AC angulum datum APM conſtituat, fiatque indeterminata AP $= x$, PM $= y$, ſit ſemper $ax = yy$ (litera *a* datam rectam deſignante) conſtat * ex hac Hypotheſi lineam curvam AMB eſſe Parabolam, cujus Diameter fuerit recta AC, Ordinata ad illam Diametrum recta PM, illius autem diametri Parameter *a*. Fig. 112. * Art. 221.

Jam ſi ponamus curvæ AMB naturam exprimi per æquationem $y^3 = aax$, vel $y^3 = axx$, illa linea curva vocatur Parabola Cubica vel Tertiæ dimensionis; ex eo quòd indeterminatarum *x* vel *y* index ma-

jor sit tertiæ dimensionis. Et si æquatio fuerit $y^4 = a^3x$, vel, $y^4 = ax^3$; curva AMB vocatur PARABOLA QUARTÆ DIMENSIONIS, quia indeterminata y cujus index omnium major sit, usque ad quartam dimensionem ascendit. Et sic in infinitum.

V.

Fig. 113. SIT, ut in definitione præcedenti, recta AC, cujus terminus A; sit BM curva ejusmodi, ut, si ex ipsius puncto quovis M agatur recta MP, quæ cum AC angulum datum APM constituat, fiatque AP $= x$, PM $= y$, sit semper $xy = aa$; (literâ a rectam datam denotante) liquet *hanc curvam fore Hyperbolam, cujus altera Asymptotos fuerit recta AC, altera vero recta AD ipsi PM parallela, & cujus dignitas vel potestas fuerit quadratum aa. Quod si æquatio, naturam curvæ BM exprimens, fuerit $xxy = a^3$, illa curva vocatur HYPERBOLA CUBICA, vel TERTIÆ DIMENSIONIS. Et si æquatio fuerit $x^3y = a^4$; curva BM erit HYPERBOLA QUARTÆ DIMENSIONIS; ex eo quod factum x^3y quatuor dimensiones habeat. Et sic de reliquis omnibus in infinitum.

* Art. 109.

COROLLARIUM I.

Fig. 112. 113. 232. SI littera m numerum quemvis integrum designet, qui index sit dignitatis vel potestatis ad quam indeterminata AP (vel x) ascendit; & si littera n denotet potestatem indeterminatæ PM, vel y, constat æquationem $y^n = x^m \times a^{n-m}$ (vel simpliciter $y^n = x^m$, factâ $a = 1$) Parabolarum omnium naturam, cujuscunque tandem dimensionis fuerint, exprimere. Eodem modo, æquationem $x^m y^n = a^{m+n}$ (vel simpliciter $x^m y^n = 1$, factâ $a = 1$) Hyperbolarum omnium naturam, quæcunque fuerint earum dimensiones, denotare. COR-

Corollarium II.

233. Si per ipsius AC terminum fixum A ducatur recta indefinita AD ipsi PM parallela; agaturque MK ipsi AC parallela, ipsi autem AD in puncto K occurrens; fiat $AK = x$; $KM = y$; liquet indeterminatam x, quæ rectam AP vel MK prius designarat, jam fieri y; & è contrario indeterminatam y, quæ PM vel AK prius designarat, jam fieri x. Unde patet,

1mo. Si curva AMB fuerit Apolloniana Parabola, eam exprimi per æquationem $yy = ax$ vel $xx = ay$, respectu habito ad rectam AC vel AD; Parabolam etiam cubicam exprimi per æquationem $y^3 = axx$, vel per $x^3 = a^2y$, respectu habito ad rectam AC vel AD. Et universaliter, si curva AMB per æquationem $y^n = x^m a^{n-m}$ exprimatur respectu habito ad rectam AC, eandem curvam denotari per æquationem $x^n = y^m a^{n-m}$ (ponitur enim n major esse quam m) respectu habito ad rectam AD. Fig. 112.

2do. Hyperbolam communem vel Apollonianam semper exprimi per æquationem $xy = aa$, respectu habito ad rectam AC vel AD; Hyperbolam vero cubicam, cujus æquatio fuerit $xxy = a^3$ respectu habito ad rectam AC, habere æquationem $xyy = a^3$, respectu habito ad rectam AD. Et universaliter, Hyperbolam, cujus æquatio respectu habito ad AC fuerit $x^m y^n = a^{m+n}$, exprimi per æquationem $x^n y^m = a^{m+n}$ respectu habito ad rectam AD. Fig. 113.

Corollarium III.

234. Hinc constat, duas esse Parabolas Cubicas, quarum una per æquationem $y^3 = aax$ vel $x^3 = aay$ exprimatur; altera vero per æquationem $y^3 = axx$ vel $x^3 = ayy$;

ayy; unicam vero Hyperbolam cubicam $xxy = a^3$ vel $xyy = a^3$. Indeterminatæ enim x & y ſolummodo quatuor primis modis componi poſſunt ad exprimendas Parabolas cubicas vel tertiæ dimenſionis; & duobus tantum modis, ad exprimendas Hyperbolas Cubicas. Jam vero, cum quatuor primæ æquationes ad duas curvas inter ſe diverſas pertineant; duæ autem poſteriores ad unam ſolummodo ſpectent; ſequitur duas eſſe Parabolas cubicas, & unam Hyperbolam cubicam. Eadem ratione inveniri poteſt numerus Parabolarum vel Hyperbolarum quartæ vel quintæ dimenſionis, &c.

Corollarium IV.

Fig. 113. 235. Rectæ indefinitæ AC, AD, non ſunt tantum Aſymptoti Hyperbolæ communis, ſed & omnium Hyperbolarum cujuſcunque dimenſionis. Si enim æquatio

$$x^m y^n = a^{m+n} \text{ vel } y^n = \frac{a^{m+n}}{x^m} \text{ (Ponitur AP} = x, \text{ PM} = y)$$

naturam cujuſvis Hyperbolæ exprimat, quando puncta ejus quoad rectam AC ſpectentur; liquet, quo major fuerit AP vel x, eo magis y^n, ac proinde PM vel y diminui; adeo ut, ſi x infinitè augeatur, PM vel y prorſus evaneſcat; hoc eſt, Hyperbola BM & recta AC in infinitum productæ magis magiſque ad invicem accedunt, nunquam tamen ſibi mutuo occurrunt, niſi in ipſo, ut
* Art. 110. ita dicam, infinitatis termino; & hæc * eſt Aſymptoti proprietas.

Jam ſi ejuſdem Hyperbolæ puncta quoad rectam AD ſpectentur; habebimus æquationem $x^n y^m = a^{m+n}$ vel $y^m = \frac{a^{m+n}}{x^n}$ (AK $= x$, KM $= y$,) Unde ſequitur, quo major

jor fit AK vel x, eo minorem fieri KM vel y, & sic in infinitum; ideoque recta AD est etiam Asymptotos ejusdem Hyperbolæ.

PROPOSITIO XIV.

PROBLEMA.

236. DETUR *punctum* M *super secundam Parabolam cubicam* AMB, *cujus natura per æquationem* $y^3 = axx$ *exprimatur*; *propositum sit tangentem* MT *ducere.* Fig. 112.

Sit arcus MN infinite parvus, ducaturque NQ ipsi PM, MR ipsi AC parallela: & triangulum MRN simile erit triangulo TPM, ex eo quod arcus exiguus MN pro tangentis productæ parte *haberi possit. * Art. 189.

Hoc posito, fiat subtangens quæsita TP $= s$, PQ vel MR $= e$; PM $= y$; & erit *RN $= \frac{ey}{s}$. * 4. El. 6.

Unde QN $=$ QR $+$ RN $= y + \frac{ey}{s}$.

Jam si loco ipsius y^3 in æquatione $y^3 = axx$, naturam curvæ AMB exprimente, substituatur cubus ipsius QN vel $y + \frac{ey}{s}$, & loco ipsius xx, quadratum ipsius AQ vel $x + e$, æquatio evadet

$y^3 + \frac{3ey^3}{s} + \frac{3eey^3}{ss} + \frac{e^3y^3}{s^3} = axx + 2eax + eea$, quæ æquatio rationem ipsius AQ ad QN perfectè exprimet.

Quod si membra prioris æquationis respectivè subducantur ex membris prioris, residuumque per e dividatur, erit

$\frac{3y^3}{s}$

$\frac{3y^3}{s} + \frac{3ey^3}{ss} + \frac{eey^3}{s^3} = 2ax + ea$. Deletis omnibus terminis, ubi e ocurrit, (existente enim PQ vel e infinitè parvâ, termini illi evanescunt) & erit tandem $\frac{3y^3}{s} = 2ax$, unde erit $PT = s = \frac{3y^3}{2ax} = \frac{3}{2}x$, substituendo, loco ipsius y^3, valorem ejus axx.

SCHOLIUM.

237. Liquet ex præcedenti calculo, si loco potestatis ipsius y substituatur eadem potentia ipsius $y + \frac{ey}{s}$, duos primos istiusce potestatis terminos solummodo requiri; reliqui enim in æquatione ultimò inventa ducti in potestates ipsius e, vel ipsam e continent, vel potestates ipsius e, ideoque prorsus sunt delendi. Idem fit, quando loco ipsius x substituatur eadem potestas ipsius $x + e$. Quod si omnes potestates radicis binomicæ $x + e$ constituas, erunt duo primi termini secundæ dimensionis $x^2 + 2ex$; tertiæ, $x^3 + 3exx$; quartæ, $x^4 + 4ex^3$; quintæ, $x^5 + 5ex^4$, & sic in infinitum; adeo ut duo primi termini potestatis cujusvis m radicis binomicæ $x + e$, sint $x^m + mex^{m-1}$. Eadem ratione constat duos primos terminos potestatis cujusvis n radicis binomicæ $y + \frac{ey}{s}$, esse $y^n + \frac{ney^n}{s}$.

Cor-

Corollarium.

238. Unde, ope æquationis generalis $y^n = x^m a^{n-m}$, vel (facta $a = 1$) $y^n = x^m$, generaliter exprimi potest subtangens PT (s) Parabolarum omnium cujuscunque dimensionis; hoc modo,

Ponantur, in æquatione generali $y^n = x^m$, loco ipsius y^n, duo primi termini potestatis n radicis $y + \frac{ey}{s}$, hoc est $y^n + \frac{ney^n}{s}$; & loco ipsius x^m ponantur duo primi termini potestatis m radicis $x + e$; hoc est, $x^m + mex^{m-1}$; unde erit $y^n + \frac{ney^n}{s} = x^m + mex^{m-1}$. Subducantur membra prioris æquationis ($y^n = x^n$) ex membris posterioris residuumque per e dividatur, & erit $\frac{ny^n}{s} = mx^{m-1}$; unde $s = \frac{ny^n}{mx^{m-1}} = \frac{nx^m}{mx^{m-1}} = \frac{n}{m}x$, nam $y^n = x^m$.

PROPOSITIO XV.

Problema.

239. Ducere *tangentes ad Hyperbolas cujuscunque dimensionis*. Fig. 113.

Manente Propositionis præcedentis constructione, ponantur, in æquatione generali $x^m y^n = a^{m+n}$ rationem ipsius AP (x) ad PM (y) denotante, loco ipsius x^m, duo primi termini potestatis m radicis AQ, ($x + e$) hoc est, $x^m + mex^{m-1}$; & loco ipsius y^n, duo primi

 termini

termini poteſtatis n radicis QN ($y-\frac{ey}{s}$) hoc eſt, $y^n-\frac{ney^n}{s}$, unde oritur hæc æquatio

$$x^m y^n + mey^n x^{m-1} - \frac{ney^n x^m}{s} - \frac{mneey^n x^{m-1}}{s} = a^{m-n}$$ rationem ipſius AQ ad QN exprimens. Membris igitur prioris æquationis $x^m y^n = a^{m+n}$ ex membris poſterioris reſpectivè ſubductis, dividendo per ey^n, erit $mx^{m-1} - \frac{nx^m}{s} - \frac{mnex^{m-1}}{s} = o$; deleto autem evaneſcente termino $-\frac{mnex^{m-1}}{s}$ ex eo quod rectam PQ (e) infinite parvam in ſe contineat, & erit ordinatis terminis PT, vel $s = \frac{nx^m}{mx^{m-1}} = \frac{n}{m}x$.

Corollarium.

Fig. 112. 113. 240. Hinc in Parabola vel Hyperbola cujuſvis dimenſionis per datum quodvis punctum M, duci poteſt tangens MT; modò æquatio pro Parabola fuerit $y^n = x^m a^{n-m}$, & pro Hyperbola, $x^m y^n = a^{m+n}$; ſumatur ſcilicet ſubtangens PT $= \frac{n}{m}$AP, ex eadem parte quà punctum A reſpectu habito ad P, quando tangens ad Parabolam ſit ducenda; ex oppoſita autem parte, quando tangens ad Hyperbolam ſit ducenda;

PRO-

PROPOSITIO XVI.

THEOREMA.

241. SIT, *ut in definitione quarta*, AMB *Parabola cujuscunque dimensionis, per æquationem* $y^n = x^m a^{n-m}$ *expressa; ducatur autem in ea ex puncto quovis* B *recta* BC, *quæ cum* AC *angulum datum* ACB *constituat, compleaturque parallelogrammum* ACBD; *dico parallelogrammum circumscriptum* ACBD *esse ad spatium Parabolicum* ACBMA *inter rectas* AC, CB *&* *Parabolæ portionem* AMB *comprehensum, sicut* $m + n$ *ad* n. Fig. 114.

Scilicet ACBD : ACBMA :: $m + n : n$.

Sumatur super Parabolæ AMB portionem arcus MN infinitè vel indefinitè parvus, hoc est minor quavis Parabolæ portione, utut exigua: ducanturque rectæ MP, NQ, ipsi BC; & MK, NL ipsi AC parallelæ; quæ exiguum parallelogrammum MRNS constituent; agatur tangens MT diametro AC in puncto T occurrens, & per T ducta parallela CB ipsis MK, NL, in punctis F, G, occurret. Hoc posito, constat arcum exiguum MN haberi * posse pro latere uno Polygoni Portionem Parabolæ AMB constituentis, tangentem vero pro latere illo producto; adeo ut duo triangula NRM, MPT, sint & rectilinea & similia, unde erit NR vel MS : RM :: MP : PT vel MF. Ergo erit Parallelogrammum PMRQ * æquale parallelogrammo FMSG; jam vero MF vel * PT $= \frac{n}{m}$ AP vel $\frac{n}{m}$ MK. Ergo etiam Parallelogrammum FMSG vel PMRQ $= \frac{n}{m}$ KMSL, & cum hoc idem fit

* Art. 189.

* 14. El. 6.

* Art. 240.

 quacun-

quacunque in Parabolæ parte cadat arcus exiguus MN, sequitur summam omnium parallelogrammorum PMRQ, hoc est, * Trilineum Parabolicum ACBMA $= \frac{n}{m}$ ADBMA summæ scilicet omnium parallelogrammorum $\frac{n}{m}$ KMSL.

* Art. 184.

Unde mACBMA $= n$ ADBMA,

* 16. El. 6. Et * ADBMA : ACBMA :: $m : n$.

Ergo ADBMA + ACBMA : ACBMA :: $m + n : n$.

Hoc est, ACBD : ACBMA :: $m + n : n$.

Corollarium I.

242. Hinc constat Trilineum Parabolicum APM esse ad Parallelogrammum circumscriptum APMK, sicut n ad $m + n$; ideoque trapezium Parabolicum MPCB $= \frac{n}{m+n}$ ABCD $- \frac{n}{n+m}$ APMK; ex eo quod ACBMA $= \frac{n}{m+n}$ ACBD, & APM $= \frac{n}{n+m}$ APMK.

Corollarium II.

243. Si Parabola data AMB fuerit Apolloniana, hoc est secundæ dimensionis, æquatio generalis $y^n = x^m a^{n-m}$ evadit $y^2 = ax$; unde in hoc casu $n = 2$, $m = 1$; ideoque

$$ACBD : ACBMA :: 3 : 2.$$

Si Parabola data esset Cubica, vel tertiæ demensionis, æquatio generalis evaderet in hoc casu $y^3 = ax^2$, ideoque ACBD

ACBD : ACBMA :: 5 : 3. Et ſic in infinitum.

PROPOSITIO XVII.

THEOREMA.

244. SIT, *ut in definitione quinta,* BMO *Hyperbola cujuſcunque dimenſionis per æquationem* $x^m y^n = a^{m+n}$ *deſignata; ducatur autem ex puncto quovis* C *recta* BC *alteri Aſympto-to* AD *parallela, & ad alteram in puncto* C *terminata, compleaturque parallelogrammum* ACBD; *dico parallelogrammum inſcriptum* ACBD *eſſe ad ſpatium Hyperbolicum* ECBMO *ſub recta determinata* BC, *ſub recta* CE *infinita verſus* C, *& ſub portione Hyperbolica* BMO *comprehenſum, ſicut* m — n : n. Fig. 115.

Scilicet ACBD : ECBMO :: $m - n : n$.

Manente præcedentis propoſitionis conſtructione, eadem ratione oſtendi poteſt, exiguum parallelogrammum PMRQ $= \frac{n}{m}$ KMSL. Et cum hoc verum ſit, quacunque in Hyperbolæ BMO parte cadat arcus MN, ſequitur ſummam omnium parallelogrammorum PMRQ, hoc eſt, *ſpatium ECBMO, æquari ſpatio $\frac{n}{m}$ EADBMO ſummæ videlicet parallelogrammorum omnium $\frac{n}{m}$ KMSL. Erit igitur *Art. 184.

EADBMO : ECBMO :: $m : n$,

Et EADBMO — ECBMO : ECBMO :: $m - n : n$.

Hoc eſt, ACDB : ECBMO :: $m - n : n$.

COROLLARIUM I.

245. HINC ſequitur trapezium hyperbolicum CPMB =

$= \frac{n}{m-n}$ ACBD $= \frac{n}{m-n}$ APMK; ex eo quod ECBMO $= \frac{n}{m-n}$ ACBD, & quod ob eandem rationem EPMO $=$ $\frac{n}{m-n}$ APMK.

Corollarium II.

246. Hinc, *1mo.* Si m major ſit quam n, ratio parallelogrammi inſcripti ACBD ad ſpatium ECBMO verſus E indefinitum, ſemper exprimitur per numeros poſitivos; unde in hoc caſu ſemper daretur abſoluta iſtiuſce ſpatii quadratura.

2do. Quando $m = n$, quod fit in Hyperbola communi vel Apolloniana, erit ACBD : ECBMO :: $o : 1$; hoc eſt ſpatium ECBMO infinitum eſt reſpectu habito ad parallelogrammum inſcriptum ABCD.

3tio. Quando m minor eſt quam n, parallelogrammum inſcriptum ACBD erit ad ſpatium hyperbolicum ECBMO, ut numerus negativus ad poſitivum; unde in hoc caſu ratio iſtiuſce ſpatii ad parallelogrammum ACBD eſt, ut ita dicam, plus quam infinita.

Notandum eſt tamen, quod in hoc poſtremo caſu ſpatium Hyperbolicum ſub recta DB, ſub Aſymptoto AD verſus D infinitâ, & ſub Hyperbola OMB comprehenſum, erit ad parallelogrammum inſcriptum ACBD ſicut m ad $n - m$, hoc eſt, ſpatii illius quadratura datur; ſi enim indeterminatæ (x) ſuper Aſymptoton AD jam ſumantur (ſumptæ enim erant antea ſuper AC) æquatio ad Hyperbolam evadet * $x^n y^m = a^{m+n}$.

* Art. 233.

PRO-

PROPOSITIO XVIII.

THEOREMA.

247. SI *sit in angulo recto* CAD *quævis curva* AMB, *cujus tangens* MT *in quovis puncto* M *datur; & si in angulo* DAH *qui deinceps sit, constituatur alia curva* HFE *ejusmodi; ut, si super eam ex puncto quovis* F *ducatur recta* FM *ipsi* AC *parallela, rectæ* AD *in* K, *curvæ autem in* M *occurrens, agaturque tangens* MT *ipsi* AC *in puncto* T *occurrens: sit semper* AK *ad* MT *sicut constans quædam* a *(eadem semper manens, quacunque in parte cadat punctum* F) *ad* KF; *dico, si per punctum quodvis* D *in recta* AD *ducatur recta* EB *ipsi* AC *parallela & ad duas curvas terminata, spatium* ADEFH *æquari rectangulo sub curva* AMB *& constanti* a *comprehenso.* Fig. 116.

Scilicet ADEFH $=$ AMB $\times\, a$.

Sumpto super curvam AMB arcu MN infinitè parvo, ductisque MF, NG ipsi AC parallelis, rectæ AD in punctis K, L, curvæ autem HFE in punctis F & G occurrentibus, agantur rectæ FS, MR ipsi AD parallelæ, producaturque RM, usque dum ipsi AC in P occurrat.

Hoc posito triangula MPT, MRN, sunt similia, unde MR : MN :: MP vel AK : MT :: q : KF.

Ergo KF $\times$ MR, hoc est, FKLS $=$ MN $\times\, a$.

Et cum hoc verum sit, quacunque in parte curvæ AMB sumatur arcus MN, sequitur summam exiguorum rectangulorum FKLS, hoc est, *spatium ADEFH æquari summæ exiguorum rectangulorum MN $\times\, a$, hoc est rectangulo sub curva AMB & constanti a comprehenso. * Art. 184.

COROLLARIUM I.

248. HINC sequitur rectangulum sub portione AM & constanti a, æquari spatio AKFH; ita & rectangulum sub portione MB & eadem recta a comprehensum æquari spatio KDEF.

COROLLARIUM II.

249. SI curva AMB fuerit secunda Parabola cubica, cujus æquatio sit $y^3 = axx$ ($AP = x$, $PM = y$;) erit * $PT = \frac{3}{2}x$. Et ob triangulum rectangulum MPT, Hypothenusa $MT = \sqrt{yy + \frac{9}{4}xx}$; verum ex natura curvæ HFE, erit $MP : MT :: a : KF$, vel $y : \sqrt{yy + \frac{9}{4}xx} :: a : KF$, Unde $\overline{KF}^2 = aa + \frac{9aaxx}{4yy}$; sed $y^3 = axx$, ergo substitutione facta erit $\overline{KF}^2 = aa + \frac{9}{4}ay$. Hinc constat, curvam HFE esse in hoc casu Parabolam, cujus axis est recta AD, vertex autem O, ita ut punctum D sit ex ex altera parte puncti A, & O ex altera, adeo ut $AO = \frac{4}{9}a$ & parameter ejus $= \frac{9}{4}a$; est enim * ex natura Parabolæ quadratum ipsius KF æquale rectangulo sub KO & parametro $\frac{9}{4}a$: hoc est, $\overline{KF}^2 = aa + \frac{9}{4}ay$. Jam vero cum trapezia Parabolica ADEH, AKFH * quadrari possint, constat dari rectificationem & curvæ AMB, & portionis cujusvis AM.

* Art. 236.

* Art. 22.

* Art. 242.

Si quæratur vera quantitas portionis AM, notandum est $AH = a$, quoniam $\overline{AH}^2 = AO \times \frac{9}{4}a = aa$; adeoque facta tangente $MT = t$, recta AK vel $MP = y$, erit

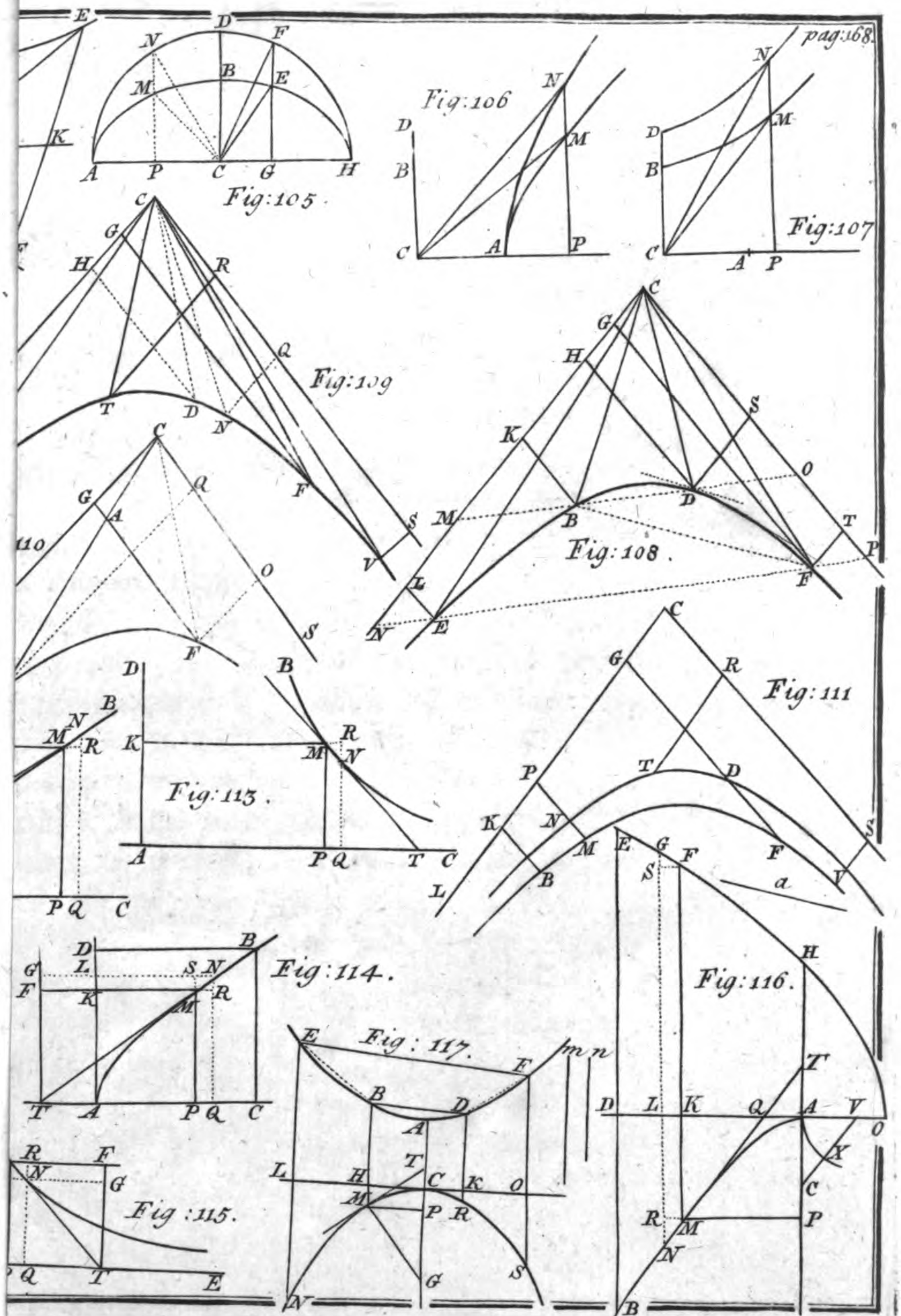
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Fig:105.
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Fig:111
Fig:113.
Fig:114.
Fig:115.
Fig:116.
Fig:117.

erit $KF = \frac{at}{y}$, & trapezium Parabolicum FKAH vel * $\frac{2}{3}$ * Art. 242.

$FK \times KO - \frac{2}{3} HA \times AO = \frac{2}{3} at + \frac{8aat}{27y} - \frac{8}{27} aa = AM \times$

a. Hoc est, dividendo per a, portio $AM = \frac{2}{3} t + \frac{8at}{27y} -$

$\frac{8}{27} a$. Unde oritur hæc constructio.

Actâ ex puncto dato M super super secundam Parabolam cubicam AMB tangente MT, quæ ipsi AK per axis AC verticem ad eundem axem perpendiculariter ductæ normalis sit, sumatur super rectam AK pars $AV = \frac{8}{27} a$, ducaturque VC ipsi MT parallela axi in puncto C occurrens; describatur centro N radio autem VA arcus Circularis AX ipsam VC in puncto X secans. Dico portionem AM secundæ Parabolæ cubicæ AMB æquari summæ rectarum MQ, CX.

Ob triangula enim TPM, TAQ similia, erit $MQ = \frac{2}{3}$ MT (t), ex eo quod $AP = \frac{2}{3}$ PT; ob triangula similia MPT, VAC, erit

$$MP : MT :: AV : VC,$$

Hoc est, $y : t :: \frac{8}{27} a : \frac{8at}{27y}$, unde $CX = \frac{8at}{27y} - \frac{8}{27} a$; sed $MQ = \frac{2}{3} MT$, ideoque $MQ + CX = \frac{2}{3} t + \frac{8at}{27y} - \frac{8}{27} a$ $= AM$.

PROPOSITIO XIX.

THEOREMA.

Fig. 117. 250. SIT EAF *Hyperbola Equilatera, cujus centrum* C, *semiaxis primus* AC; *sit etiam* NCS *Parabola, cujus axis fuerit recta* AC *versus Verticem* C *producta, axis autem Parameter dupla* CA. *Jam si per quodvis in Parabola* NCS *punctum* N, *agatur recta* NE *ipsi* CA *parallela, Hyperbolæ ipsi* EAF *in puncto* E, *ejus autem axi secundo* CL *in puncto* L *occurrens; dico spatium Hyperbolicum* CLEA *sub rectis* AC, CL, LE, *& sub Hyperbolæ portione* EA *comprehensum, æquari rectangulo sub Parabolæ portione* CN, *& recta* AC *contento.*

Scilicet $CLEA = CN \times AC$.

Ducatur per punctum quodvis M in portione Parabolica CN recta MG tangenti MT per illud punctum ductæ, & ad axem in T terminatæ perpendicularis, ad axem etiam in G terminata; agatur MB ipsi CA parallela, Hyperbolæ in puncto B, secundo autem axi CL in puncto H occurrens; & erunt rectæ MG, HB, sibi invicem æquales. Ducta enim ad axem Ordinata MP,
* Art. 28. erit * $PG = CA$; & ob triangulum rectangulum MPG,
* Art. 140. erit etiam $\overline{MG}^2 = \overline{PM}^2 + \overline{PG}^2 = \overline{CH}^2 + \overline{CA}^2 = {}^*\overline{HB}^2$, ob Hyperbolam equilateram EAF; unde $MG = HB$. Triangula autem TPM, MPG sunt similia, unde MP vel
* Art. 246. CH : MT :: PG vel CA : MG vel HB. Ergo * spatium Hyperbolicum $CLEA = CN \times AC$.

COROLLARIUM II.

251. HINC constat, trapezium Hyperbolicum HLEB

æquari

æquari rectangulo sub portione Parabolica MN, & semiparametro CA axis.

Corollarium II.

252. Si in Hyperbola equilatera EAF agantur duæ parallelæ BD, EF, ducanturque per earum extremitates rectæ BM, EN, DR, FS, ipsi AC parallelæ, secundo axi Hyperbolæ in punctis H, K, L, O, occurrentes, differentia rectangulorum $AC \times MN$, $AC \times RS$ æquabitur differentiæ trapeziorum rectilineorum HLEB, KOFD.

Nam rectangulum $AC \times MN$ *æquatur trapezio Hyperbolico HLEB, ideoque rectangulum $AC \times MN +$ segmentnm Hyperbolicum $EB =$ trapezio rectilineo HLEB; eadem ratione rectangulum $AC \times RS +$ segmentum Hyperbolicum $DF =$ trapezio rectilineo KOFD; sed segmenta EB, DF *æquantur; ergo differentia rectangulorum $AC \times MN$, $AC \times RS$ æquabitur differentiæ trapeziorum rectilineorum HLEB, KOFD.

*Art. 250.

*Art. 208.

Corollarium III.

253. Iisdem, quæ in consectario præcedenti, positis; si fiat $2AC : LH :: \overline{BH + LE} : m$; liquet rectangulum $AC \times m = \frac{1}{2} LH \times \overline{BH + LE}$, hoc est trapezio rectilineo HLEB; eadem ratione, si fiat $2AC : KO :: KD + FO : n$, erit $AC \times n =$ trapezio KOFD; ideoque differentia rectangulorum $AC \times RS$, æquabitur differentiæ rectangulorum $AC \times m$, $AC \times n$; hoc est, dividendo per AC, differentia arcuum Parabolicorum MN, RS, æquabitur differentiæ rectarum m, n; unde constat inveniri posse rectas æquales differentiæ arcuum Parabolicorum MN, RS, ad infinitum.

FINIS.

Corrigenda.

Pag l. 6. lege *dimidio*. p. 40 l. 16. lege $\frac{2abyy}{sx}$ pro $\frac{2abyy}{xx}$ p. 70. l. ult. leg. *æquari*. p. 88. l. ult. pro *Ch* lege CH. p. 101. l. 2. leg. *illam*. p. 151. l. 18. leg. *numerorum*. p. 141. l. 7. pro ZPY leg. SPZ. p. 143. l. 1. lege *Diametro*.

p. 128. l. 14: pervento, Lege, pervecto, Seu, provecto